A Brief introduction to the SM

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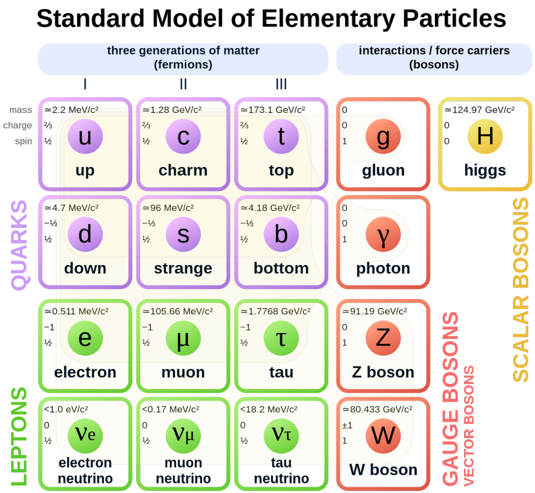

Figure: Particle content of the SM. Taken from [?]

- Interactions: Electromagnetism, Strong Force and Weak Force. Arise from the symmetries of the Lagrangian.
- As the theory implies high energies $\rightarrow$ Lagrangian must be Lorentz invariant.
- Quarks and Leptons are fermions described by spinors $\rightarrow$ Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \tag{1}$$

- How to study the symmetries of the system? Transforming the fields!

Quantum Electrodynamics (QED)

- First we consider global transformations for charged fermions (parameters do not depend on the position)

$$\psi \to e^{-iq\theta}\psi \quad \& \quad \bar{\psi} \to e^{iq\theta}\bar{\psi} \tag{2}$$

  q is the electric charge.
- By replacing these expressions into 1 one finds that the Lagrangian remains invariant.
- Noether's Theorem: There is a conserved charge $\rightarrow$ Electric charge. Noether current is

$$J^{\mu} = iq\alpha\bar{\psi}\gamma^{\mu}\psi \tag{3}$$

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- We need to check for local invariance → Transformation depends on the position

$$\begin{aligned} \psi \rightarrow e^{iq\theta(x)}\psi =&(1 + iq\theta(x))\psi \\ \bar{\psi} \rightarrow e^{-iq\theta(x)}\bar{\psi} =&(1 - iq\theta(x))\bar{\psi} \end{aligned} \tag{4}$$

- Lagrangian is not invariant as

$$\partial_\mu \psi' = e^{iq\theta(x)}(iq\partial_\mu\theta)\psi + e^{iq\theta}\partial_\mu\psi \tag{5}$$

$$\neq e^{iq\theta}\partial_\mu\psi \tag{6}$$

- To eliminate the extra term we introduce the covariant derivative

$$D_\mu = \partial_\mu + iqA_\mu, \tag{7}$$

- A_μ is called a gauge field. I.e: Field that mediates a certain physical process.
- As our fields are electrically charges A_μ=photon!
- In order for $\mathcal{L}$ to be invariant, the photon field must transform as

$$A_\mu \to A_\mu - \frac{1}{q}\partial_\mu \theta(x) \tag{8}$$

- Including the covariant derivative, the invariant Lagrangian is

$$\begin{aligned}\mathcal{L} =& i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi && (9)\\ =& i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + iq\bar{\psi}\gamma^\mu \psi A_\mu && (10)\end{aligned}$$

- Note that we obtained the free Lagrangian plus an interaction term which describes how charged fields interact via the EM field!
- After field quantization, one observes that phenomena, like electric repulsion and attraction, are just charged particles interchanging photons!

Feynman Rules and Diagrams

- According to the interaction picture of QM we can calculate all contributions to the S Matrix as

$$S(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4 x_1 \cdots \int \mathrm{d}^4 x_n T\{\mathcal{H}_i(t_1) \cdots \mathcal{H}_i(t_n)\} \quad (11)$$

- A new question arises: How do we calculate the T products? $\rightarrow$ Wick's theorem.
- One can show that the time ordered product of two fields $\phi(x)$ and $\phi(y)$ depends on the normal ordered product and the propagator:

$$T[\phi(x)\phi(y)] =: \phi(x)\phi(y) : + iD_F(x, y) \quad (12)$$

- As example let us consider one of the first terms of the S matrix

$$S^{(1)} = \exp\left[- ie \int \mathrm{d}^4 x T(\bar{\psi}^{(+)} \gamma^\mu A_\mu^{(+)} \psi^{(+)}) \right] \tag{13}$$

- In momentum space, the component that just includes annihilation operators becomes

$$S_{fi}^{(1)} = -iq \int \mathrm{d}^4 x e^{-i(p+q+k)} \bar{v}_s(p) \gamma_\mu \epsilon_r^\mu(k) u_t(q) \tag{14}$$

- After performing the integral it reduces to

$$S_{fi}^{(1)} = -iq(2\pi)^4 \delta(p+q+k) \bar{v}_s(p) \gamma_\mu \epsilon_r^\mu(k) u_t(q) \tag{15}$$

- Delta function ensures momentum conservation!
- One can rewrite the matrix element as:

$$S_{fi} = \delta(p + q + k)\mathcal{M}_{fi} \tag{16}$$

- Note that if $\mathcal{M}$ is known, the S matrix element can be easily obtained after momentum conservation.
- Feynman: There is no need to calculate the whole set of integrals to obtain both matrices $\rightarrow$ Feynman rules

- In addition to the rules, one can assign a diagram to each contribution to $\mathcal{M} \rightarrow$ Feynman diagrams
- For each interaction there is a set of rules and diagrams!

Feynman rules of QED

- For every incoming fermion assign a spinor $u_s(p)$ and the following diagram

- For every incoming antifermion assign a spinor $\bar{v}_s(p)$

- For every incoming boson assign a polarization vector $\epsilon_r^\mu(p)$

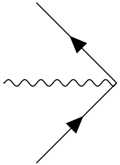
- For every outgoing fermion assing a spinor $\bar{u}_s(p)$

- For every outgoing antifermion assign a spinor $v_s(p)$

- For every outgoing boson assign a polarization vector $\epsilon^*_{\mu r}(p)$

- All particles connect into a vertex. This vertex has a momentum space function given by $-ie\gamma^\mu$

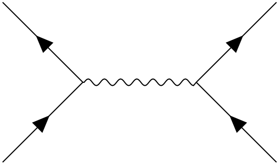
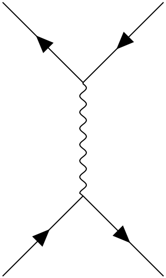
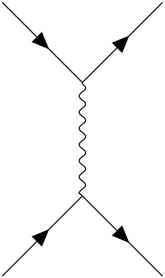
- According to these rules, the first order process that we were studying can be diagrammatically established by



$$\sim \quad \overline{\psi}^{(+)}\gamma^{\mu}A_{\mu}^{(+)}\psi^{(+)} \quad \propto \quad \overline{v}_s(p)\epsilon_{\gamma}^{\mu}(k)\gamma_{\mu}u_t(q)$$

- All matrix elements associated to first order processes vanish due to momentum conservation.
- Basic processes are second order in the S matrix expansion!

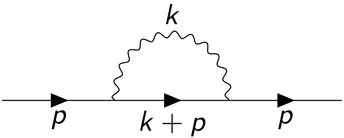
- Some examples of second order processes are

 ;  ; 

- There are internal fermion and boson lines! These also have a Feynman rule
- For each internal fermion line assign a propagator $iS_F(q)$

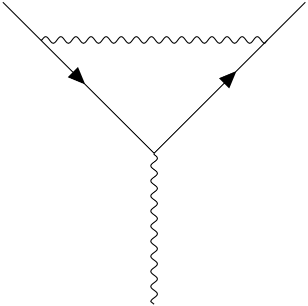
- For each internal photon line assign a propagator $iD_{\mu\nu}(q)$.

- Another type of processes are those involving loops, which are similar to the following diagram



- Loops are considered to be corrections to certain physical parameters such as the vertex function.
- The Feynman rule for loops is: For each loop present in a diagram, integrate over the momenta of the additional particle $\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$.
- Integrals arising from loop processes diverge $\rightarrow$ renormalization

- One of the most fundamental results from QED arises from loop processes! Namely, the correction to the magnetic moment of the electron



- The magnetic moment of the electron, g, obtains a correction given by

$$\left(\frac{g-2}{2}\right)_{theo} = 0.001159652411 \quad (17)$$

- It matches remarkably with the experimental results up to twelve significant figures

$$\left(\frac{g-2}{2}\right)_{exp} = 0.001159652209 \tag{18}$$

- QED is a very precise theory!

The Strong Interaction: Quarks and QCD

- QED was the first successful QFT. However it does not explain the remaining 3 fundamental forces: Strong, Weak and Gravitation.
- So far, there was no understanding of the physics that governs the atomic nucleus.
- 1940 $\rightarrow$ Heisenberg, Jordan and others started to study the force that binds the nucleus together, the Strong force.

- 1932: Neutron was first discovered by Chadwick.
- 1934: Planck, Weizsäcker and Bethe made progress in understanding, empirically, how the nucleus behaves.
- Fundamental result: The strong interaction does not distinguish between protons and neutrons!
- Heisenberg: One can say that a quantum state for a nucleon lives in a 2D complex Hilbert space

$$|N\rangle = a\,|p\rangle + b\,|n\rangle \tag{19}$$

Strong Isospin

- Symmetries of this Hilbert space are given by SU(2)$\rightarrow$ Isospin (strong).
- We choose p and n to be eigenstates of T^2 and T_3

$$T_3 \left| p \right\rangle = \frac{1}{2} \left| p \right\rangle ; T_3 \left| n \right\rangle = -\frac{1}{2} \left| n \right\rangle \tag{20}$$

- Hence, protons and neutrons form an isospin doublet.

- For isospin, we have the same Lie group as in QM spin, so we look for the Isospin operator

$$\vec{T} = T_1\hat{x} + T_2\hat{y} + T_3\hat{z} \tag{21}$$

- If we remember our basic quantum mechanics, our isospin operators must obey the Lie Algebra

$$[T_i, T_j] = i\epsilon_{ijk} T_k \tag{22}$$

- Any choose of T's following the SU(2) algebra, will then give us a representation of it. The lowest dimensional irreps are proportional to the Pauli matrices

$$T_i = \frac{1}{2}\tau_i \tag{23}$$

- The generators of the group elements are the Pauli matrices which are given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{24}$$

- In general, for any SU(N) group we have

$$\#\text{bosons} = \#\text{generators} = N^2 - 1 \tag{25}$$

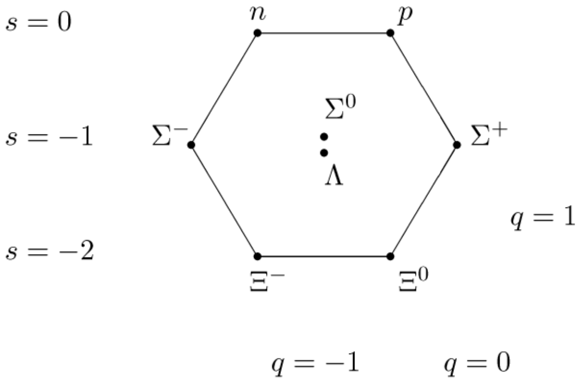
- Experimental confirmation $\rightarrow \eta^0$, $\pi^\pm$ and π^0 discovery.
- Strong Isospin theory was quickly disregarded after the discovery of Δ resonances $\rightarrow$ New quantum number "Strangeness S"
- Protons, neutrons and $\Delta \rightarrow$ Baryons while Pions $\rightarrow$ Mesons
- To distinguish between them a new quantum number was introduced $\rightarrow$ Baryon Number (B)
- Zweig: It is possible to unify S and B in a single number $\rightarrow$ Strong Hypercharge

$$Y = B + S \tag{26}$$

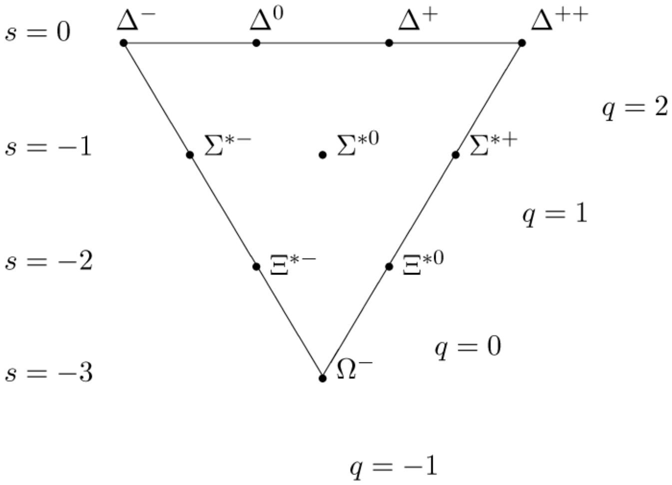
- One then could classify particles in tables, or multiplets. For example, we can arange all baryons in an octet

Part.	T	T_3	B	Y	S
p	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
n	$\frac{1}{2}$	-$\frac{1}{2}$	1	1	0
Λ^0	0	0	1	0	-1
Σ^+	1	1	1	0	-1
Σ^0	1	0	1	0	-1
Ξ^0	$\frac{1}{2}$	$\frac{1}{2}$	1	-1	-2
Ξ^-	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	-2

- The baryon octet could be then geometrically represented by



- More particles were discovered that matched with the Isospin, Hypercharge and Strangeness scheme. One example are those that can be aranged in a decuplet



- Gell-Man and Nishijima: We can recover the electric charge of strong interacting particles via

$$Q = T_3 + \frac{1}{2}Y \tag{27}$$

- Gell-Man: The octets and decuplets take the form of the weight diagrams for SU(3)
- As SU(3) is the group of transformations of a 3D complex Hilbert space, the canonical basis vectors must be

$$|u\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |d\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |s\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{28}$$

- They called these vectors quarks.

- Up, down and strange quarks (and their antiparticles) live in the irreducible fundamental representation of SU(3)
- All particles in multiplets could be formed from these quarks. Here, it became possible to note

$$\text{Baryons} \to |qqq\rangle \qquad \text{Mesons} \to |q\bar{q}\rangle \tag{29}$$

- SU(3) flavor symmetry predicts that resonances that Δ and Ω violate Pauli exclusion principle as

$$\left|\Omega^{-}\right\rangle = |sss\rangle \qquad \left|\Delta^{++}\right\rangle = |uuu\rangle \tag{30}$$

The SU(3) Color symmetry

- Quark model predicts new hadrons that were later discovered. However, it breaks Pauli exclusion principle.
- 1965 Yoshiro Nambu and Greenberg proposed that Isospin, strangeness, hypercharge, charmness, beauty and topness are not fundamental quantum numbers.
- Introduced a new SU(3) fundamental symmetry, which carries a charge exclusive to quarks: color charge.
- In this sense, quarks can be described by

$$|\psi\rangle = C_i \, |\psi(p)\rangle \tag{31}$$

where C_i is a color vector (or superposition of them), and $|\psi(p)\rangle$ is a Dirac spinor.

- Color vectors have the form

$$C_1 = r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad C_2 = g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad C_3 = b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{32}$$

- So existence of resonances can be explained such that each quark carries a certain color, such that the final hadron is neutral.
- We need to build a QFT with SU(3) invariant Lagrangians $\rightarrow$ QCD

SU(3)

- SU(3) Lie Algebra takes a familiar form

$$[T_i, T_j] = if_{ijk} T_k \quad (33)$$

Note that we have different structure constants than for SU(2).

$$f^{123} = 1$$
$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$$
$$f^{458} = f^{678} = \frac{\sqrt{3}}{2},$$

- The generators are not the same ones as for SU(2)

- The 8 generators are given in terms of the Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- Hence, any element os SU(3) can be written as $g \in SU(3) = \exp(i\alpha_a T_a)$

- We have that λ_3 and λ_8 are diagonal matrices. This means that we can tag our states with their possible values. Namely, t_3 and Y:

$$\{|t_3, y\rangle\}\text{forms a basis.} \tag{34}$$

- This allows us to define the hypercharge operator as

$$Y = \frac{2}{\sqrt{3}}T_8 = \frac{1}{\sqrt{3}}\lambda_8 \tag{35}$$

- Note that if there were no overlaping between the SU(2) subalgebras, the flavor representation would made sense.
- In other words, the Hypercharge-Isospin symmetry arises from SU(3) Casimir operators! However, the algebra overlaping makes it non exact.

- How to construct our multiplets from a possible set of states?$\rightarrow$ Young Tableaux
- We will say that quark states transform in the fundamental representation, call it 3, whereas out anti-quarks are in the anti-fundamental one, call it $\bar{3}$.
- Mesons $\rightarrow |q\bar{q}\rangle$. So we need to find all possible representations given by the product $3 \otimes \bar{3}$
- Using Young Tableaux theory, one can find that

$$3 \times \bar{3} = 8 \oplus 1 \tag{36}$$

- This means that mesons can live in an octet and a singlet. In terms of flavour physics: Octet is the one we saw before, the singlet is represented by the η meson

- In color theory: Octet represents the set of forms on which colors in gluons can be arranged, while the singlet has proven to be unphysical as gauge boson while it explains the uncolored state of mesons
- Color combinations are also given by

$$g_1 = r\bar{g}; g_2 = g\bar{r}; g_3 = r\bar{b}; g_4 = b\bar{r}; g_5 = g\bar{b}$$
$$g_6 = b\bar{g}; g_7 = \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}); g_8 = \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$$

- Now we need to build an SU(3) invariant QFT

QCD

- Quarks are spin half particles $\rightarrow$ Dynamics must be governed by the usual Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \tag{37}$$

- From our last seminar, we know that these spinors must carry color indices to preserve SU(3)
- Flavor indices are also introduced to reduce the number of terms in the Lagrangian

$$\mathcal{L} = i\bar{\psi}^f_c\gamma^{\mu}\partial_{\mu}\psi^f_c - m\bar{\psi}^f_c\psi^f_c \tag{38}$$

- Redefine the spinors as

$$\bar{\psi}^f = (\bar{\psi}^f_r, \bar{\psi}^f_g, \bar{\psi}^f_b) \tag{39}$$

- Thus

$$\mathcal{L} = i\bar{\psi}^f \gamma^\mu \partial_\mu \psi^f - m\bar{\psi}^f \psi^f \tag{40}$$

- Global SU(3) transformations (recall that $T_a = \frac{1}{2}\lambda_a$ and $a = 1, \cdots, 8$)

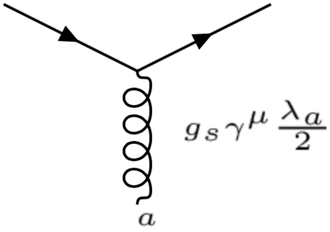
$$\psi_f \rightarrow e^{i\alpha_a T_a}\psi_f \qquad \bar{\psi}_f \rightarrow \bar{\psi}_f e^{-i\alpha_a T_a} \tag{41}$$

- As the α_a are space-time independent, the Lagrangian is invariant under these global transformation.
- Noether current is then given by

$$J_a^\mu = \bar{\psi}_f \gamma^\mu T_a \psi_f \tag{42}$$

note that there is one conserved charge for each color combination.

- This gives us our first Feynman rule for the quark-gluon vertex



- Now we introduce local transformations

$$\psi_f \to e^{ig_s\alpha_a(x)T_a}\psi_f \qquad \bar{\psi}_f \to \bar{\psi}_f e^{-ig_s\alpha_a(x)T_a} \tag{43}$$

Hence

$$\partial_\mu\psi_f \to \partial_\mu\psi_f + ig_s(\partial_\mu\alpha_a)T_a\psi_f \tag{44}$$

- Extra term is eliminated if we introduce the SU(3) covariant derivative

$$D_\mu = \partial_\mu + ig_s T_a G^a_\mu \tag{45}$$

- The gauge field G_μ must then transform as

$$G^a_\mu \to G^a_\mu - \partial_\mu \alpha_a - g_s f_{abc} \alpha_b G^c \mu \tag{46}$$

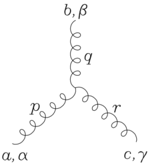
- Then, we define the Field strength tensor in a similar way than for the photon field

$$F^{\mu\nu}_a = \partial^\mu G^\nu_a - \partial^\nu G^\mu_a + g_s f_{abc} G^\mu_b G^\nu_c \tag{47}$$

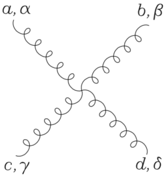
- Gauge invariant Lagrangian reads as

$$\mathcal{L} = i\bar{\psi}_f \gamma^\mu \partial_\mu \psi_f - m\bar{\psi}_f \psi_f - g_s J^\mu_a G^a_\mu - \frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} \tag{48}$$

- From this Lagrangian one can read the remaining tree level Feynman rules



$$= \; g f^{abc}\left[g^{\alpha\beta}(p-q)^{\gamma} + g^{\beta\gamma}(q-r)^{\alpha} + g^{\gamma\alpha}(r-p)^{\beta}\right]$$



$$\begin{aligned} = \; & -ig^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ & -ig^2 f^{xad} f^{xbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ & -ig^2 f^{xab} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned}$$

- We can also introduce propagators for the quark fields, as they can be quantized as any fermion field

$$\delta_{ab} \frac{i(\gamma^{\mu} p_{\mu} + m)}{p^2 - m^2 + i\epsilon}$$

- However, we are not done! Usually, quantization of gauge fields becomes problematic as irredundant degrees of freedom must be taken out of the theory.
- For Yang-Mills theories, this quantization becomes really messy and requires the introduction of a new formalism: Path Integrals. In this formalism, the irredundancies can be taken out by introducing Ghost fields and the Faddeev-Popov determinant (out of the scope of this talks).

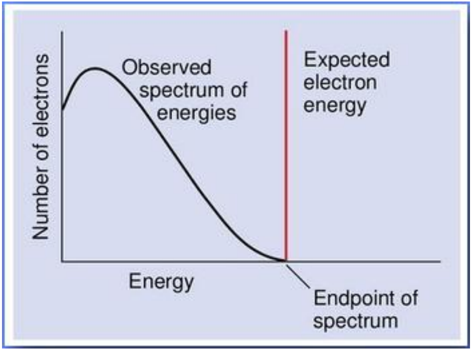
- In addition to quantization, path integrals become really useful when studying renormalization and lattice theories. Here, Kenneth Wilson discovered that any SU(3) gauge theory is confining.
- Confinement implies that one can have a short-range interaction with massless fields, such that they produce a potential

$$V(r) = \frac{a}{r} + br \tag{49}$$

- This means that quarks can not be observed as free particles at low energies!

Experimental problems

- Spectrum of emitted electrons expected to be at just one value (Δm).
- Experiments had shown that electron spectrum is continous, it ranges from m_0 to Δm
- Energy apparently is not being conserved in this processes.

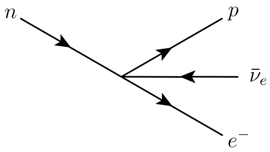


Solutions

- Böhr: energy conservation is a statitical phenomena such as entropy.
- Pauli: new neutral particle called neutron(later neutrino) so the missing energy is that of this particle.
- Solvay conference 1933: neutrino should be a fermion(more specifically a lepton) so it can satisfy conservation laws.

Fermi Theory

- Beta decay is a weak interaction as it's spacial range is little compared to the other.
- Weak currents describe the process, J_μ^{Had},J_μ^{Lep}
- Currents are connected to the same spacial vertex.
- Strenght of the interaction is given by a constant G_F



Problems of Fermi Theory

- 1956: Yang and Lee noted that there was no experimental evidence of parity conservation in weak processes
- 1957: Wu discovered that the ^{60}Co decay violates parity!
- Weak interaction distinguishes between Left Handed(LH) and Right Handed chiralities !
- Solution given by Feynman: Introduce V-A currents such that LH components of the spinor participate in the interaction, whilst the RH are singlets!

$$J^{\mu} = \frac{G_F}{\sqrt{2}}\bar{\psi}\gamma^{\mu}P_L\psi; \quad P_L = \frac{1}{2}(1-\gamma_5) \tag{50}$$

- Using these (V-A) currents, obligates us to use a new representation of the gamma matrices, namely the Weyl (or chiral) representation, where they are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{51}$$

- Chirality $\rightarrow$ eigenstates of the γ_5 matrix
- As γ_5 is diagonal, one can decompose a Dirac spinor

$$\psi = \psi_L + \psi_R \tag{52}$$

- In terms of the chiralities, Free Dirac Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R - m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R \qquad (53)$$

- Note that an SU(2) transformation like

$$\psi_L \to \psi_L' \propto \exp(i\alpha\gamma_5)\psi_L$$

  will not leave the mass terms invariant, we say then that mass terms break gauge invariance. Additionally, note that as neutrinos only participate in weak processes, they can only posses left-handed chirality which means that they are massless.
- Short range interaction implies the existence of massive vector mediators!

Weak Interactions and Isospin

- In terms of quarks, the β decay can be understood as $d \to u + e^- + \bar{\nu}_e$
- Let us analyze the properties of d and u

Quark	Spin	Mass(MeV)	Charge
d	1/2	4.7	-1/3
u	1/2	3.3	2/3

- If mass is set to zero (gauge invariance) and we turn off the electric charge, these two particles are basically the same.
- Pauli exclusion principle allows us to understand u and d as basis vectors of a 2D complex Hilbert space (same as with spin up and down):

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{54}$$

Doublets and currents

- As mentioned before, Wu's experiment implies that the left handed components of u and d form a doublet

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{55}$$

- Same happens with leptons

$$\ell_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \tag{56}$$

in this case e$=e^-$, μ^- or τ^-.

- Take any of these doublets to be ψ. The dynamics of the doublet are given by

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \tag{57}$$

Our duty is now to study the symmetries of this Lagrangian.

- There are in fact two different symmetry transformations
  1. ψ as a doublet has an SU(2) transformation.
  2. Each individual component of ψ, together with their right-handed components, acts as a Dirac spinor $\rightarrow$ U(1) transformation.
- We focus on the SU(2) transf.

$$\psi \rightarrow e^{-ig_w \tau_a \theta_a}\psi \tag{58}$$

$$\bar{\psi} \rightarrow \bar{\psi}e^{ig_w \tau_a \theta_a} \tag{59}$$

Here we used Einstein's index summation rule and take a=1,2,3.

- If θ_a are space-time independent, we have a global symmetry and the Lagrangian is invariant.
- One Noether current for each value of a

$$J_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = g_w \bar{\psi} \gamma^\mu \tau_a \psi \tag{60}$$

- If we take ψ to be a lepton doublet, we have that the currents are

$$J_1^\mu = g_w(\bar{\nu}_e \gamma^\mu e + \bar{e} \gamma^\mu \nu_e) \tag{61}$$
$$J_2^\mu = -i g_w(\bar{\nu}_e \gamma^\mu e - \bar{e} \gamma^\mu \nu_e) \tag{62}$$
$$J_3^\mu = g_w(\bar{\nu}_e \gamma^\mu \nu_e - \bar{e} \gamma^\mu e) \tag{63}$$

- Note that J_3 resembles an electromagnetic current (first glimpse to unification)

$$J^{\mu}_{EM} = Q\bar{e}\gamma^{\mu}e \tag{64}$$

- Now, we want our physics to be local

$$\psi \to e^{-ig_w\tau_a\theta_a(x)}\psi \tag{65}$$

$$\bar{\psi} \to \bar{\psi}e^{ig_w\tau_a\theta_a(x)} \tag{66}$$

- Kinetic term is non invariant as

$$\partial_\mu\psi' = -ig_w(\tau_a\partial_\mu\theta_a)e^{-ig_w\tau_a\theta_a}\psi + e^{-ig_w\tau_a\theta_a}\partial_\mu\psi \tag{67}$$

- Hence

$$\mathcal{L}' = i\bar{\psi}\gamma^{\mu}\partial_\mu\psi + g_w\bar{\psi}\gamma^{\mu}(\tau_a\partial_\mu\theta_a)\psi \tag{68}$$

- Extra term breaks gauge invariance. We require to introduce the SU(2) covariant derivative

$$D_\mu = \partial_\mu + i g_w \tau_a W^a_\mu \tag{69}$$

- W_1, W_2 and W_3 are the vector bosons associated to the gauge connection coefficients, and therefore will mediate the interaction.
- With this introduction, the lagrangian becomes

$$\mathcal{L} = i\bar{\psi}'\gamma^\mu D_\mu \psi' \tag{70}$$

$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi - g_w\bar{\psi}e^{i\tau_a\theta_a}\gamma^\mu\tau_b(W_{b\mu} + \partial_\mu\theta_b)e^{-i\tau_c\theta_c}\psi \tag{71}$$

- One can use the Baker-Campbell-Hausdorff formula to show

$$e^{i\tau_a\theta_a}\tau_b e^{-i\tau_c\theta c} = \tau_b + i\sum_b \theta_a\epsilon_{abc}\tau_c \tag{72}$$

- From now on we replace indices to i, j, k.

- The lagrangian becomes

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}D_{\mu}\psi + 2i\bar{\psi}\gamma^{\mu}\sum_{ijk}\theta_i W_{\mu j}\tau_k\epsilon_{ijk}\psi \tag{73}$$

- To vanish the extra term we require the W's to transform in the adjoint representation. This means

$$W_{i\mu} \to W_{i\mu}\partial_{\mu}\theta_i - 2g_w\sum_{jk}\epsilon_{ijk}W_{j\mu}\theta_k \tag{74}$$

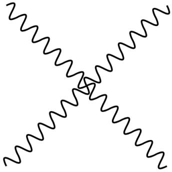
- With this transformation of the Lagrangian, it will remain invariant. However, we still do not have a term for W boson kinematics, hence we introduce

$$F^i_{\mu\nu} = \partial_{\mu}W^i_{\nu} - \partial_{\nu}W^i_{\mu} + 2g_w\sum_{jk}\epsilon_{jk\ell}W_{\mu k}W_{\nu\ell} \tag{75}$$

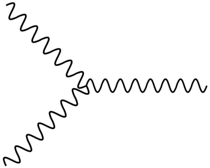
- Kinetic Lagrangian is then

$$\mathcal{L}_w = -\frac{1}{4}F_{\mu\nu i}F^{\mu\nu}_i \tag{76}$$

- This lagrangian induces 4 boson terms with the likes of $W_\mu^a W_\nu^b W_\alpha^c W_\beta^d$. In diagramatic form this looks like



- 3 boson terms are also possible via $(\partial_\mu W_a^\mu) W_\alpha^b W_\beta^c$



- Total lagrangian is now

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - J_\mu^a W_a^\mu - \frac{1}{4}F_{\mu\nu a}F_a^{\mu\nu} \tag{77}$$

- Problem: Experiment tells us that the W bosons are massive! They will be introduced via Higgs mechanism.

The $W^{\pm}$ bosons

- Recall spin: Given a basis where S^2 and S_3 are diagonal, it is possible to introduce ladder operators

$$S_{\pm} = S_1 \pm iS_2 \tag{78}$$

- As we chose an spherical basis for Isospin, we can build ladder operators:

$$T_{\pm} = T_1 \pm T_2 \tag{79}$$

- This allows us to build vector bosons for each of these operators, while preserving the number of degrees of freedom:

$$W_{\mu}^{\pm} = \frac{1}{\sqrt{2}}(W_{\mu}^{1} \pm iW_{\mu}^{2}) \tag{80}$$

with Noether currents given by

$$J_{\pm}^{\mu} = \frac{1}{2}(J_{1}^{\mu} \pm iJ_{2}^{\mu}) \tag{81}$$

- Replacing the expressions gives us two charged currents

$$J_{+}^{\mu} = g_w \bar{\nu}_L \gamma^{\mu} e_L \quad (82)$$

$$J_{-}^{\mu} = g_w \bar{e}_L \gamma^{\mu} \nu_L \quad (83)$$

- They also have their own strength tensor

$$F_{\mu\nu}^{\pm} = F_{\mu\nu}^{(1)} \mp iF_{\mu\nu}^{(2)} \quad (84)$$

Neutral currents

- Theory has an additional symmetry which arises from the Dirac behavior of the spinors, just like they do in QED!
- Extra $U(1)_Y$ symmetry transforms the fields via

$$\psi_L \to \exp\left(-\frac{i}{2}g' Y_L \theta\right)\psi_L \tag{85}$$

$$\psi_R \to \exp\left(\frac{i}{2}g' Y_R \theta\right)\psi_R \tag{86}$$

- LH and RH spinors are not forced to have the same hypercharge values, this is due to their doublet and singlet behavior under $SU(2)_L$.

- Following the same procedure than in QED, we obtain two new neutral currents

$$J_{\mu}^{Y_L L/R} = \frac{g'}{2} Y_{L/R} \bar{\psi}_{L/R} \gamma^{\mu} \psi_{L/R}, \qquad (87)$$

- Also the covariant derivative is

$$D_{\mu}^{L/R} = \partial_{\mu} + i g' Y_{L/R} B_{\mu}, \qquad (88)$$

- The presence of the new $U(1)_Y$ gauge symmetry, together with $SU(2)_L$, in the weak interaction means that the total gauge group is $SU(2)_L \times U(1)_Y$.

Unification

- For left-handed leptons, the neutral part of the Lagrangian reads

$$\mathcal{L} \supset \frac{-1}{2}[(g' Y_L B^\mu + g_w W_3^\mu)\bar{\nu}_L \gamma^\mu \nu_L + (g' Y_L B^\mu - g_w W_3^\mu)\bar{e}_L \gamma_\mu e_L] \tag{89}$$

- Note that the terms between parenthesis add in a similar way as the result of a matrix multiplication with a vector. In other words, it takes the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_{11} a + a_{12} b \\ a_{21} a + a_{22} b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

- Glashow, Weinberg, Salam: Structure of the interaction take a form similar to that of QED, namely $QA^{\mu}J_{\mu} \rightarrow$ Electromagnetism and Weak interactions are manifestations of a single interaction!
- Introduce a new gauge boson Z_{μ} such that W_3^{μ} and B_{μ} transform to

$$\begin{pmatrix} B^{\mu} \\ W_3^{\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^{\mu} \\ Z^{\mu} \end{pmatrix}, \tag{90}$$

- Z boson was discovered in the 1970's

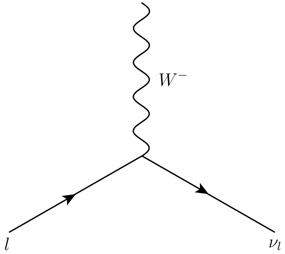
- The total Lagrangian will now be

$$\begin{aligned}\mathcal{L} =& i\psi\gamma^{\mu}\partial_{\mu}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}W_{\mu\nu}W^{\mu\nu} - J^{\mu}_{\pm}W^{\pm}_{\mu} \\ & - Q\bar{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{g_w}{2\cos\theta_w}\bar{\psi}\gamma^{\mu}(c_v - c_a\gamma_5)\psi Z_{\mu}\end{aligned} \quad (91)$$

- We can obtain the Feynman rules of the Electroweak theory!

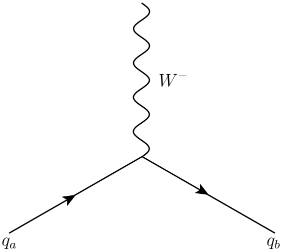
Feynman Rules of EW theory

- Lepton-Neutrino coupling with W boson



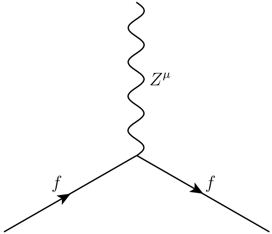
$$-i\frac{g_W}{\sqrt{2}}\gamma^\mu P_L$$

- Quark coupling with W boson



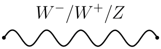
$$-\,i\frac{g_W}{\sqrt{2}}V_{ab}P_L$$

- Fermion coupling with the Z boson



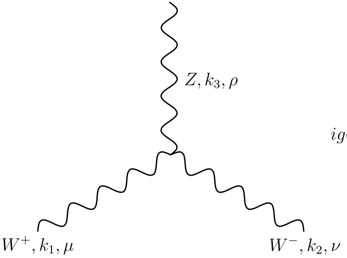
$$-i\frac{g_W}{2\cos\theta_W}\gamma^\mu(C_v - C_a\gamma_5)$$

- Vector boson propagator



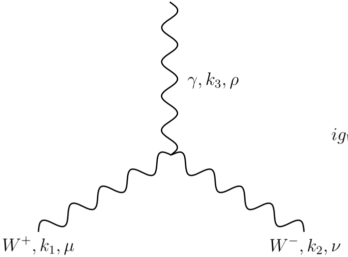
$$iD_{\mu\nu}(p) = -i\frac{(\eta_{\mu\nu} - p_\mu p_\nu/m^2)}{p^2 - m^2 + i\epsilon}$$

- W boson coupling with Z boson



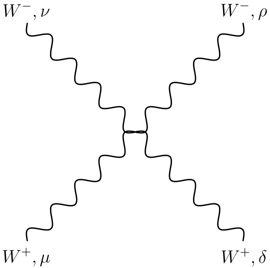
$$ig_W \cos\theta_W [\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_2 - k_3)_\mu + \eta_{\rho\mu}(k_3 - k_1)_\nu]$$

- W boson coupling with the photon



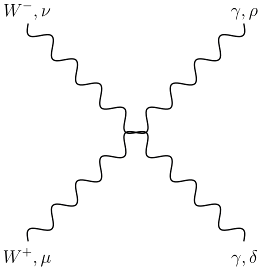
$$ig_W \sin\theta_W [\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_2 - k_3)_\mu + \eta_{\rho\mu}(k_3 - k_1)_\nu]$$

- 4 vector boson coupling



$$ig_W^2[2\eta_{\mu\delta}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\delta\rho} - \eta_{\mu\rho}\eta_{\nu\delta}]$$

- W boson-Photon 4 point interaction



$$-ig_W^2 \sin^2\theta_W[2\eta_{\mu\nu}\eta_{\delta\rho} - \eta_{\mu\rho}\eta_{\nu\delta} - \eta_{\mu\delta}\eta_{\nu\rho}]$$

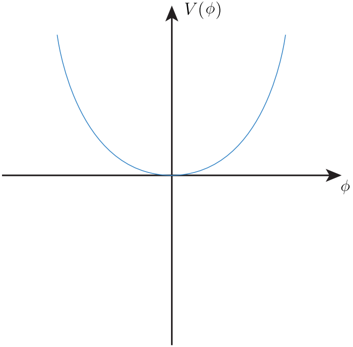
The Higgs Mechanism

- We know that mass terms explicitly break the SU(2) symmetry $\rightarrow$ Needs to be recovered
- Solution was found by Englert, Kibble, Brout and Higgs!
- Introduce a scalar field doublet with non trivial interaction term

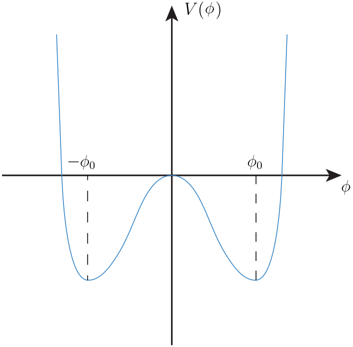
$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 \tag{92}$$

Spontaneously Symmetry Breaking

- If $\mu^2 > 0$, the potential is symmetric and the symmetry is preserved such that the minimum is at $\phi_0 = 0$



- However, if $\mu^2 < 0$ the symmetry breaks such that the potential changes its form



- Multiple minima occurring at $|\phi_0|^2 = \frac{-2\mu^2}{\lambda}$
- As the base of the potential is, topologically speaking, a circle no extra energy is required for moving around it, introducing redundancies known as Goldstone bosons.
- As we are choosing a minimum, we can expand the doublet in terms of angular and radial modes (Goldstone and Higgs fields respectively) as

$$\phi = \begin{pmatrix} G^{\pm} \\ \frac{v+h+iG^0}{\sqrt{2}} \end{pmatrix} \tag{93}$$

- To preserve the local $SU(2) \times U(1)$ gauge invariance of the doublet Lagrangian, we must introduce the covariant derivative

$$D_\mu = \partial_\mu + ig_w T_a W_\mu^a + i\frac{g'}{2} B_\mu, \tag{94}$$

- Replacing derivatives induces quadratic terms for the vector bosons!
- How to deal with the Goldstone bosons? $\rightarrow$ Unitary gauge $\rightarrow$ Vector bosons "eat" these modes and become massive!
- In this gauge, the doublet becomes

$$\phi = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \tag{95}$$

- To give mass to fermions we need to introduce an interaction term with the scalar doublet → Yukawa

$$\mathcal{L}_{Yuk} = \Gamma^{u}_{ij}\bar{q}_{Li}\tilde{\phi}u_{Rj} + \Gamma^{d}_{ij}\bar{q}_{Li}\phi d_{Rj} + \Gamma^{e}_{ij}\bar{e}_{Li}\phi e_{Rj}; \quad \tilde{\phi} = -i\tau_2\phi^* \tag{96}$$

- Γ matrices are the couplings between fermions and the Higgs doublet
- Without loss of generality, let us consider only one lepton family with Yukawa Lagrangian given by

$$\mathcal{L} \supset f_e\bar{e}_L\phi e_R \tag{97}$$

- After SSB the Lagrangian becomes

$$\mathcal{L} \supset f_e \bar{e}_L \left(\frac{v+h}{\sqrt{2}} \right) e_R \tag{98}$$

$$= \frac{f_e v}{\sqrt{2}} \bar{e}_L e_R + \frac{f_e}{\sqrt{2}} \bar{e}_L h e_R \tag{99}$$

- By noticing that the first term is a constant coupling both chiral states, we define the lepton mass as $m_e = \frac{f_e v}{\sqrt{2}}$, such that the Lagiangian becomes

$$\mathcal{L} \supset m \bar{e}_L e_R + \frac{m}{v} \bar{e}_L h e_R \tag{100}$$

- We obtained fermion masses!
- By considering leptons as singlets under SU(3), we can construct a general gauge group for the theory is

$$SU(3)_c \times SU(2)_L \times U(1)_Y \tag{101}$$

- This is known as the Standard Model of particle physics!