(Mini) Introduction to cosmological perturbation theory

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Cosmological perturbations

Summary

1. Inflation. Background evolution and basic definitions.
2. Scalar field inflation.
3. Inflationary perturbations and gauge fixing.
4. Stochastic properties and statistical approach.
5. Cosmological perturbations in large scale structure formation.

Notation

Natural units: $c = h = 1.$

Signature: $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)\,.$

Cosmological Background

Einstein equations and Friedman-Lemaitre-Robertson-Walker (FLRW) solution

Einstein equations dictates the dynamics of the universe:

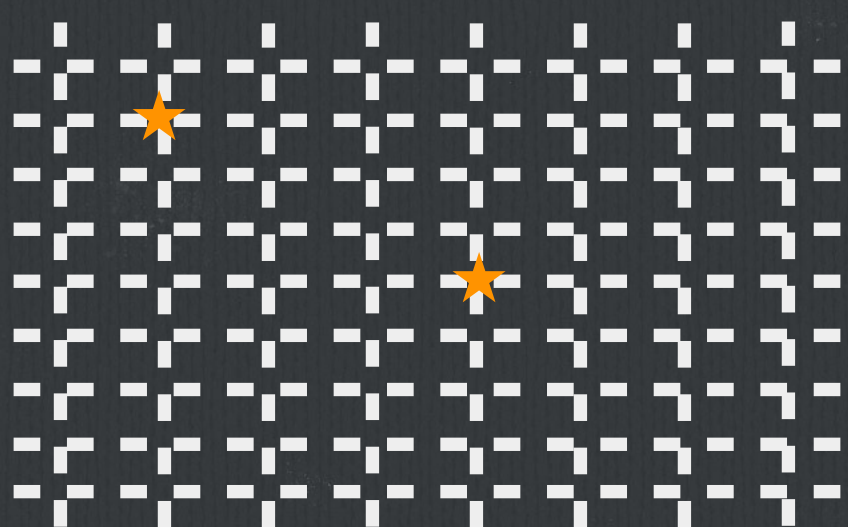
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} .$$

$$R^{\sigma}{}_{\mu\rho\nu} = \partial_\rho \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\alpha\rho}\Gamma^\alpha_{\mu\nu} - \Gamma^\sigma_{\alpha\nu}\Gamma^\alpha_{\mu\rho}, \quad \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\alpha g_{\mu\nu}\right) .$$

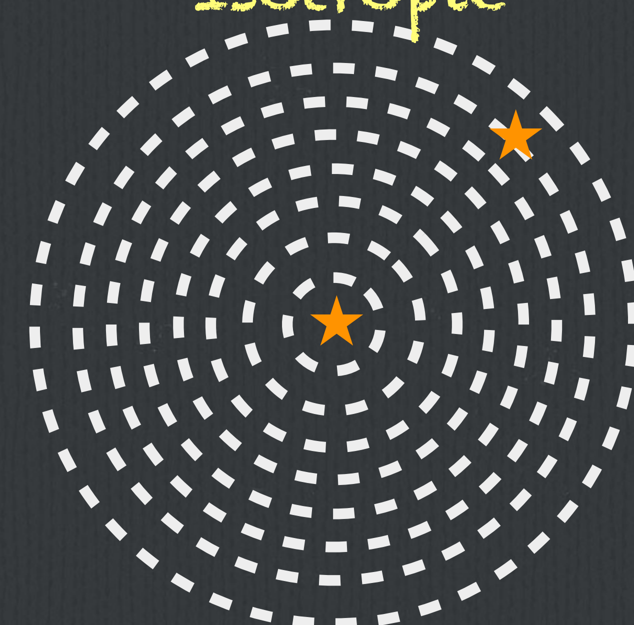
$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}, \quad R = g^{\mu\nu} R_{\mu\nu} .$$

Cosmological principle: homogeneous and isotropic fluid at large scales. L ~ 1 - 100 Mpc. 1 Mpc = 3×10^{24} cm ~ 3×10^{6} - 3×10^{8} light-years.

Homogeneous



Isotropic



Cosmological Background

Einstein equations and Friedman-Lemaitre-Robertson-Walker (FLRW) solution

FLRW solution: A solution reflecting isotropy and homogeneity

$$ds^2 = -\,dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j\,.$$

Symmetric under translations and rotations:

$$x^i \to x^i + d^i, \quad x^i \to R^i{}_j x^j,\ \ R^i{}_j \in SO(3)\,.$$

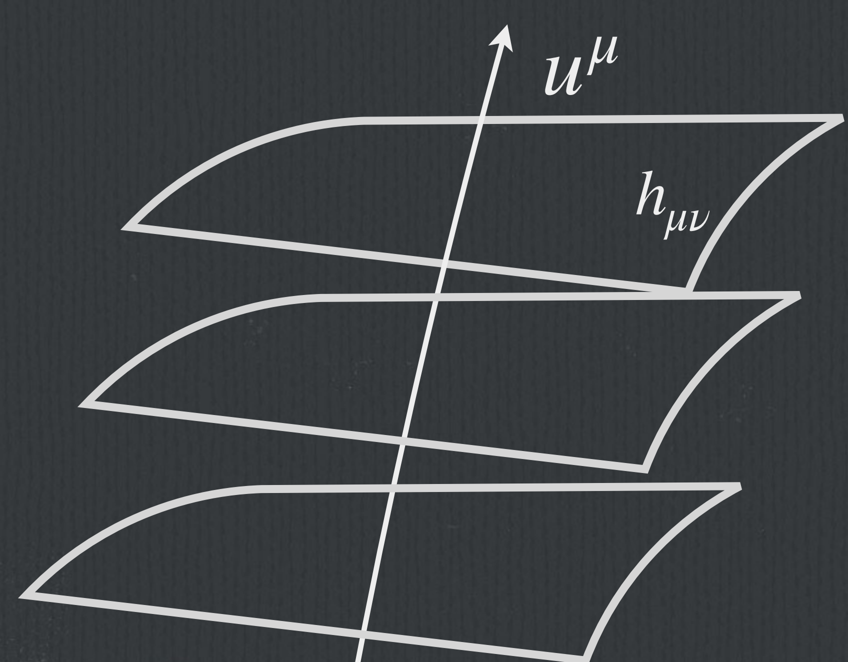
Kinematic quantities:

Observeres (timelike) 4-velocity

$$u^\mu = \frac{dx^\mu}{ds}, \quad g_{\mu\nu}u^\mu u^\nu = -\,1, \quad u^\mu = (\partial_t)^\mu = (1,0,0,0)\,.$$

Induced spacial metric:

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad h_{\mu\nu}u^\mu = 0.$$



Cosmological Background

Einstein equations and Friedman-Lemaitre-Robertson-Walker (FLRW) solution

Matter contents: Energy momentum tensor of a general fluid:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \Sigma_{\mu\nu} \,.$$

$\rho = T_{\mu\nu}u^\mu u^\nu \rightarrow$ Matter-Energy density.

$p = \frac{1}{3}T_{\mu\nu}h^{\mu\nu} \rightarrow$ Isotropic pressure.

$q_\mu = -h_\mu{}^\sigma T_{\sigma\rho}u^\rho \rightarrow$ Energy flux

$\Sigma_{\mu\nu} = h_{(\mu}{}^\sigma h_{\nu)}{}^\rho T_{\sigma\rho} \rightarrow$ Zero trace anisotropic stress tensor

Energy momentum tensor of a single, perfect fluid ($q_\mu = 0, \Sigma_{\mu\nu} = 0$):

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \quad T_{\mu\nu} = \text{diag}(\rho, p, p, p) \,.$$

Continuity equation

$$\nabla^\mu T_{\mu\nu} = 0.$$

Cosmological Background

Einstein equations and Friedman-Lemaitre-Robertson-Walker (FLRW) solution

Friedmann equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,.$

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right), \quad R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right).$$

Hubble rate

$$H \equiv \frac{\dot{a}}{a}, \qquad H^2 = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p)\,.$$

Continuity equation

$$\nabla^{\mu} T_{\mu\nu} = 0 \quad \rightarrow \quad \frac{d\rho}{dt} + 3H(\rho + p) = 0.$$

Equation of state (barotropic fluid)

$$p = \omega\rho\,.$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}3\rho(1+\omega) = 0 \quad \rightarrow \quad \frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a} + 3(1+\omega) = 0 \quad \rightarrow \quad \boxed{\rho = \rho_0 a^{-3(1+\omega)}}\,.$$

Cosmological Background

Causal structure at large scales

Evolution of the scale factor

$$\rho = \rho_0 a^{-3(1+\omega)}, \quad H^2 = \frac{8\pi G}{3}\rho \quad \rightarrow \quad a = a_0 t^{\frac{2}{3(1+\omega)}} .$$

Particular cases of the equation of state

Ultra non-relativistic matter $\omega = 0, \quad \rho = \rho_0 a^{-3}, \quad a = a_0 t^{2/3} .$

Radiation $\omega = 1/3, \quad \rho = \rho_0 a^{-4}, \quad a = a_0 t^{1/2} .$

Vacuum, Λ $\omega = -1, \quad \rho = \rho_0, \quad a = a_0 e^{Ht} .$

Light geodesics and particle horizon

Photons trajectory $0 = \mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \rightarrow \mathrm{d}r = \frac{\mathrm{d}t}{a(t)} .$

Physical distance that a photon travels

$$R_H(t) = a(t)\int_0^t \frac{\mathrm{d}t'}{a(t')} .$$

Particle horizon

Maximum distance that an observer can reach in a causal way.

Cosmological Background

Causal structure at large scales

Comoving and physical scales

Comoving distance x. Physical distance $x_p = ax$,

Conformal time τ

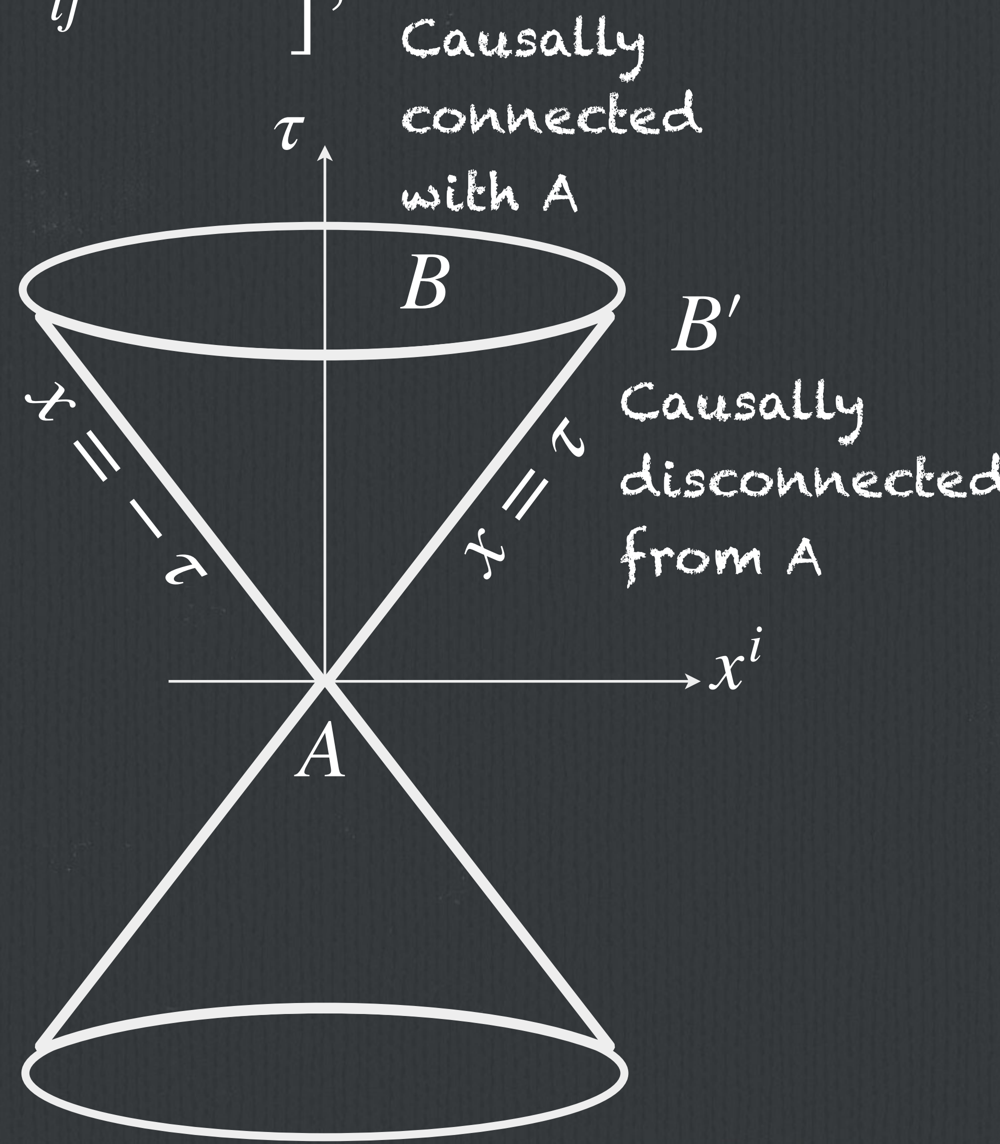
$$ds^2 = -\,dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j = a^2(\tau)\left[-d\tau^2 + \delta_{ij}dx^i dx^j\right],$$

$$\Rightarrow \quad \boxed{d\tau = \frac{dt}{a(t)},\quad \tau = \int_0^t \frac{dt'}{a(t')}\,.}$$

Null geodesics

$$x = \pm\,\tau + \text{const}\,.$$

Frontiers between regions in causal contact with an observer, and regions without causal contact.



Cosmological Background

Causal structure at large scales

Hubble radius

$$\tau = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^t \frac{1}{a(t')} \frac{\mathrm{d}t'}{\mathrm{d}a} \mathrm{d}a = \int_0^a \left(\frac{1}{aH} \right) \mathrm{d} \ln a \,.$$

Comoving particle horizon τ.

Hubble radius: H^{-1}.

Comoving Hubble radius: $(aH)^{-1}$.

Scales and Hubble radius

Physical wavelength $\lambda = a\lambda_{\mathrm{comoving}} = a2\pi/k$

Comparison of scales: $$2\pi \frac{(aH)^{-1}}{\lambda} = \frac{k}{aH} \,.$$

$$\frac{k}{aH} \ll 1 \quad \rightarrow \quad \lambda \text{ outside the horizon} \,.$$

$$\frac{k}{aH} \gg 1 \quad \rightarrow \quad \lambda \text{ inside the horizon} \,.$$

Number of e-folds

$$\mathrm{d}N = H\mathrm{d}t = \mathrm{d}\ln a \,.$$

Cosmological Background

Inflation and the horizon problem (in few words)

Surface of last scattering and causally connected regions. The time when photons decouple from matter, at temperature T ~ 0.3 eV, at time t ~ 3x10⁵ years.

$$\tau = \int_0^a \left(\frac{1}{aH}\right) \mathrm{d}\ln a \propto a^{\frac{1}{2}(1+3\omega)}\,.$$

For $\omega > 0$, the horizon is growing monotonically, so, photons that we see today, tracked back in the past, come from regions causally disconnected. The problem with that is that those causally disconnected regions share very similar features, for instance, the distribution of temperature perturbations around decoupling time is homogeneous and isotropic. This is, causally non-communicating regions share basically the same features.

In terms o the Hubble radius. If the Hubble radius grows monotonically, then, homogeneous regions that we see today were far outside the Hubble radius in the past

$$R_H = \left(\frac{1}{H}\right)$$

Cosmological Background

Inflation and the horizon problem (in few words)

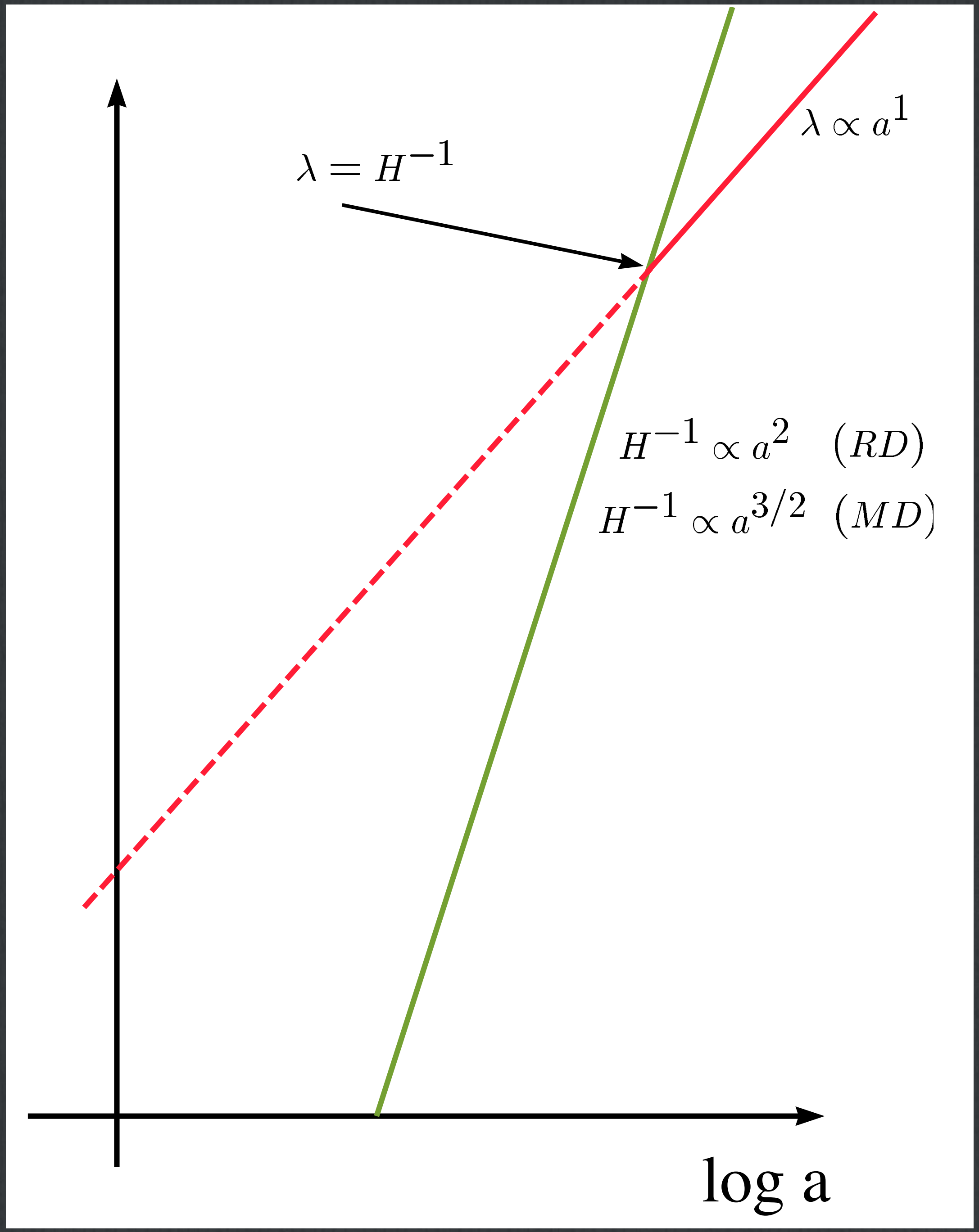
Surface of last scattering and causally connected regions.

$$r_H = \left(\frac{1}{H}\right)$$

The evolution of Hubble radius vs the evolution of a scale λ, the distance between two photons coming from last scattering surface

$$\lambda < r_H = \frac{1}{H} \quad \rightarrow \quad \lambda \text{ inside the horizon}.$$

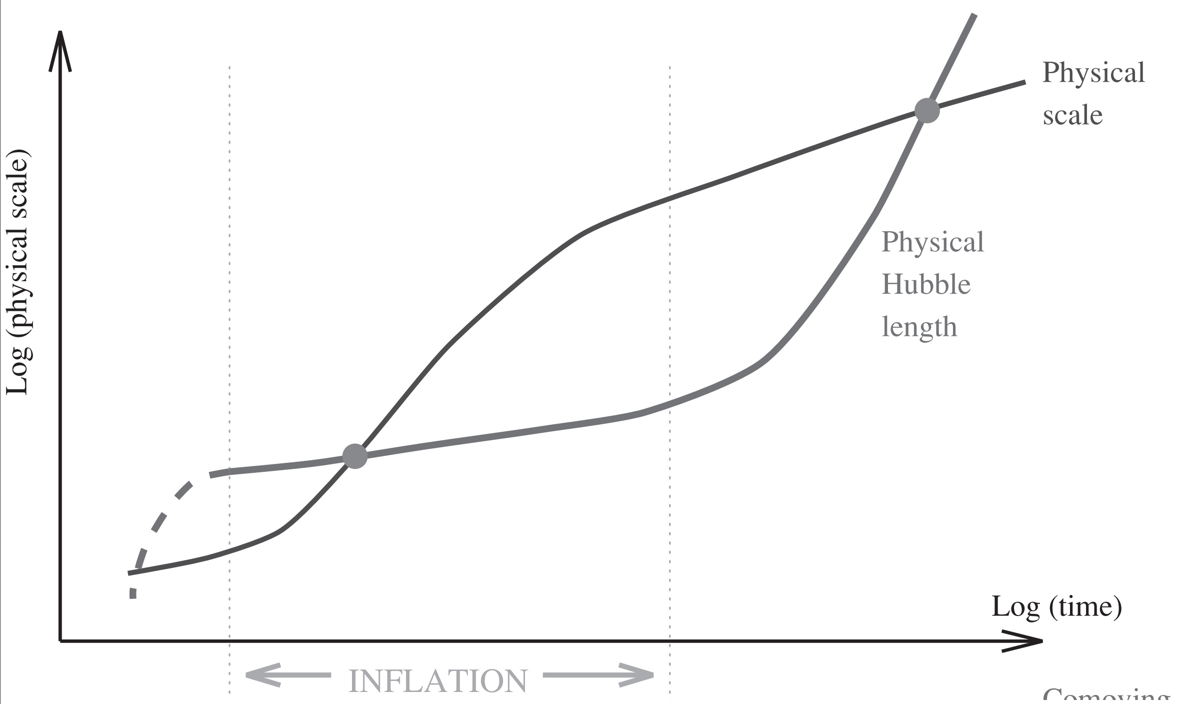
$$\lambda > r_H = \frac{1}{H} \quad \rightarrow \quad \lambda \text{ outside the horizon}.$$

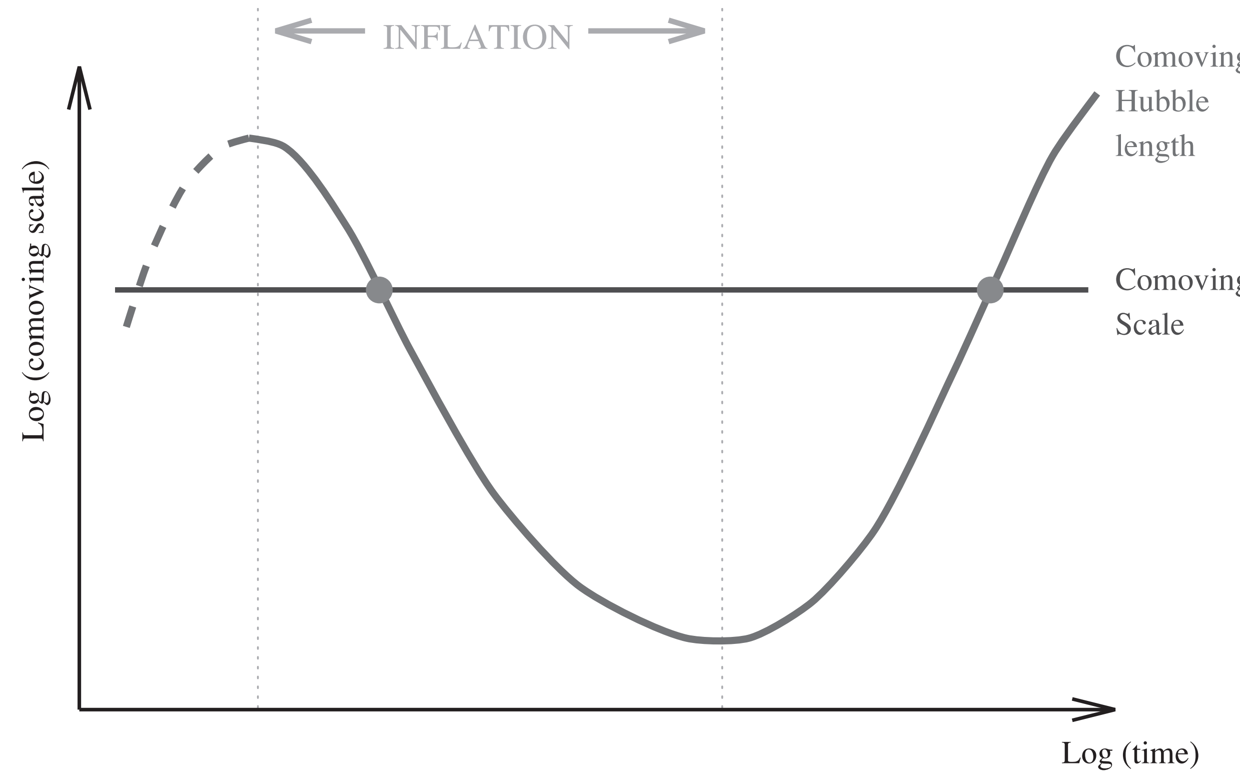


Cosmological Background

Inflation and the horizon problem (in few words)

Surface of last scattering and causally connected regions.





Hubble radius

$$r_H = \left(\frac{1}{H}\right)$$

Comoving Hubble radius

$$R_H = \left(\frac{1}{aH}\right)$$

Cosmological Background

Inflation and the horizon problem (in few words)

Inflation and the horizon problem. In order to solve the problem with causality posed by the horizon problem, we should have a shrinking Hubble radius in the past, so:

$$\frac{\mathrm{d}}{\mathrm{d}t}R_H < 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \quad \rightarrow \quad \ddot{a} > 0.$$

From Friedmann equations:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \quad \Rightarrow \quad (\rho + 3p) < 0 \quad \Rightarrow \quad \omega < -\frac{1}{3}.$$

Inflation defined in terms of the evolution of the scale factor.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \quad \Leftrightarrow \quad \ddot{a} > 0.$$

Single field inflation

Inflation driven by a single scalar field

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right]$$

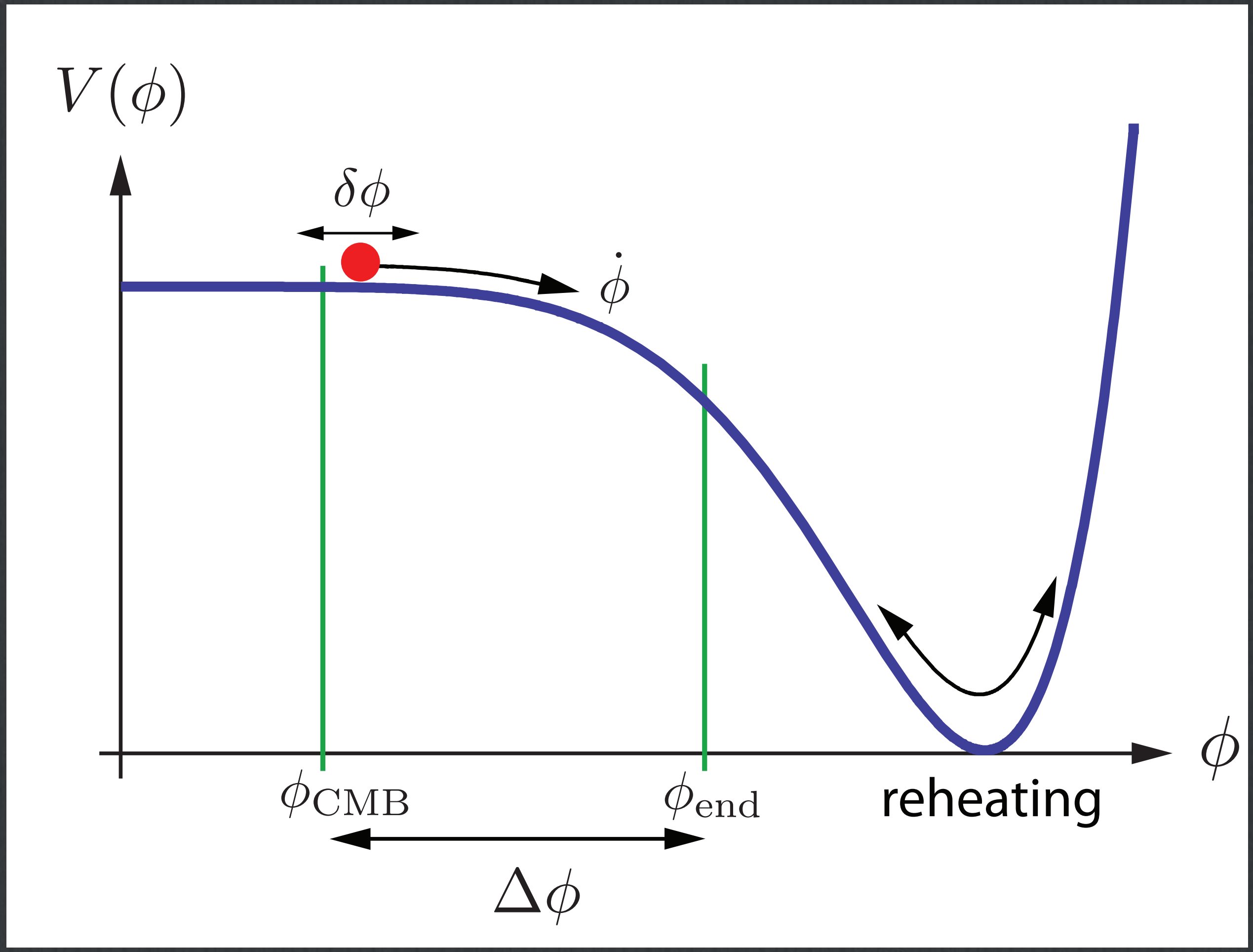


Illustration of an inflaton potential

Single field inflation

Inflaton dynamics

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right]$$

Equation of motion for ϕ

$$\frac{\delta S}{\delta\phi}: \quad \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \partial_\phi V = 0.$$

Energy-Momentum tensor

$$-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}: \quad T_{\mu\nu}^{(\phi)} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right).$$

Gravitational field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}^{(\phi)} \quad \rightarrow \quad H^2 = \frac{8\pi G}{3}\rho_\phi, \quad \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho_\phi + 3p_\phi).$$

Single field inflation

Inflaton dynamics

Homogeneous field ϕ. A homogeneous scalar field $\phi(x,t) = \phi(t)$ behaves like a perfect fluid and support the inflationary evolution

$$\ddot{\phi} + 3H\dot{\phi} + \partial_{\phi} V = 0.$$

Pressure, energy density and continuity equation

$$T_{00}^{(\phi)} = \rho_{\phi} = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$

$$T_{ii}^{(\phi)} = p_{\phi} = \frac{1}{2}\dot{\phi}^2 - V(\phi),$$

$$\frac{d\rho_{\phi}}{dt} + 3H\rho_{\phi}(1 + \omega_{\phi}) = 0.$$

$$\omega_{\phi} = \frac{p_{\phi}}{\rho_{\phi}} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}.$$

Gravitational field equation/ Friedman equations

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(1 + 3\omega_{\phi})\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right).$$

Single field inflation

Slow roll inflation

Definition of slow roll parameter ϵ

$$\frac{\ddot{a}}{a} = -\frac{1}{2}H^2(1+3\omega_\phi) = H^2(1-\varepsilon), \quad \varepsilon \equiv \frac{3}{2}(1+\omega_\phi) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} = -\frac{\dot{H}}{H^2}.$$

$$\boxed{\varepsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N}.}$$

Accelerated expansion in terms of slow roll parameter

$$\boxed{\ddot{a} > 0 \quad \Leftrightarrow \quad \varepsilon < 1}$$

de Sitter limit $\omega_\phi \to -1,\ \varepsilon \to 0.$

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \to -1 \quad \to \quad V(\phi) \gg \dot{\phi}^2.$$

Definition of second slow roll parameter η. $\ddot{\phi}$ is small enough to sustain accelerated expansion

$$|\ddot{\phi}| \ll \{3H\dot{\phi}, \partial_\phi V\}. \quad \boxed{\eta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}N}}$$

Single field inflation

Slow roll inflation

Potential slow roll parameters

Using Friedmann equations, slow roll parameters can also be seen as conditions on the shape of the potential

$$\epsilon_V \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2, \quad \eta_V \equiv M_{\rm pl}^2 \frac{V_{\phi\phi}}{V}. \qquad M_{\rm pl}^2 \equiv \frac{1}{8\pi G}.$$

Relations between both set of parameters

$$\varepsilon \approx \epsilon_V, \qquad \eta = \eta_V - \epsilon_V.$$

Exponential expansion

$$\dot\phi \approx -\frac{V_\phi}{3H}, \quad H^2 \approx \frac{8\pi G}{3}V(\phi) \approx \text{const}. \rightarrow \frac{\dot a}{a} = H \rightarrow a \approx a_0 e^{Ht}.$$

End of inflation and number of e-folds

$$\varepsilon \approx \epsilon_V(\phi_{\rm end}) = 1 \quad \rightarrow \quad N(\phi) = \ln\left(\frac{a_{\rm end}}{a}\right) = \int_t^{t_{\rm end}} H dt = \int_\phi^{\phi_{\rm end}} H \frac{d\phi}{\dot\phi}$$

$$N(\phi) = \int_\phi^{\phi_{\rm end}} H \frac{{\rm d}\phi}{\dot\phi} \approx -8\pi G \int_\phi^{\phi_{\rm end}} \frac{V}{V_\phi}{\rm d}\phi \approx \frac{1}{M_{\rm pl}} \int_{\phi_{\rm end}}^{\phi} \frac{{\rm d}\phi}{\sqrt{2\epsilon_V}}.$$

Enough inflation to solve IC problems

$$N_{\rm tot} = \ln\left(\frac{a_{\rm end}}{a_{\rm start}}\right) \gtrsim 60$$

Inflationary perturbations

Scalar, vector and tensor degrees of freedom

Matter perturbations and metric perturbations

The Einstein equations couple matter perturbations to the metric perturbations

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}^{(\phi)} \,.$$

$$\delta\phi \rightarrow \delta T_{\mu\nu} \rightarrow \delta g_{\mu\nu} \rightarrow \delta\phi \,.$$

$$\phi(x,t) = \phi^0(t) + \delta\phi(x,t)$$

Inflationary perturbations

Cosmological Perturbations

We can study perturbations around a homogeneous background:

$$\phi(x,t) = \phi(t) + \delta\phi(x,t), \qquad g_{\mu\nu}(x,t) = g_{\mu\nu}(t) + \delta g_{\mu\nu}(t)$$

$$ds^2 = -(1+2\Phi)dt^2 + 2aB_i\, dx^i dt + a^2\left[(1-2\Psi)\delta_{ij} + E_{ij}\right] dx^i dx^j$$

Scalar, vector, tensor decomposition:

- $B_i = \partial_i B - S_i, \quad \partial^i S_i = 0$
- $E_{ij} = 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}, \quad \partial^i F_i = 0, \quad \partial^i h_{ij} = 0.$

Coordinate transformation: $t \to t + \alpha, \qquad x^i \to x^i + \delta^{ij}\beta'_j$ Scalar metric and matter transformations:

$$\Phi \to \Phi - \dot\alpha, \quad B \to B + a^{-1}\alpha - a\dot\beta$$

$$E \to E - \beta, \quad \Psi \to \Psi + H\alpha$$

$$\delta\rho \to \delta\rho - \dot\rho\alpha, \quad p \to \delta p - \dot p\alpha$$

Inflationary perturbations

Gauge invariant variables

Gauge invariant variables.

Curvature perturbation on uniform-density hypersurfaces

$$-\zeta \equiv \Psi + \frac{H}{\dot{\rho}}\delta\rho \approx \Psi + \frac{H}{\dot{\phi}}\delta\phi \quad \text{(Slow-roll)}$$

Comoving curvature perturbation:

$$\mathcal{R} \equiv \Psi - \frac{H}{\rho + p}\delta q \approx \Psi + \frac{H}{\dot{\phi}}\delta\phi \quad \text{(Slow-roll)}$$

$$-\zeta = \mathcal{R}$$

for slow-roll and on super horizon scales $k \ll aH$.

We can calculate statistical properties in the form of correlation functions (power spectrum, bispectrum, etc.) of these gauge variables!

Inflationary perturbations

Correlation Functions. Statistical Properties of Cosmological Perturbations.

Power spectrum.

$$< \mathcal{R}_k \mathcal{R}_{k'} >= (2\pi)^3 \delta(k + k') P_{\mathcal{R}}(k), \qquad \Delta^2_{\mathcal{R}} \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k)$$

$$n_s - 1 \equiv \frac{d \ln \Delta^2_{\mathcal{R}}}{d \ln k}, \qquad \alpha_s \equiv \frac{dn_s}{d \ln k}$$

Power law spectrum

$$\Delta^2_{\mathcal{R}} = A_s(k_*) \left(\frac{k}{k_*} \right)^{n_s(k_*) - 1 + \frac{1}{2}\alpha_s(k_*) \ln(k/k_*) + \ldots}$$

Inflationary perturbations

Scalar Perturbations.

Scalar action.

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]$$

Expanding up to 2nd order in $\mathcal{R}$, (this is a long exercise of integration by parts) we get:

$$S_{\text{(2nd order)}} = \frac{1}{2}\int d^4x\, a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right]$$

Mukhanov action

Defining $v \equiv z\mathcal{R}$, $\quad z \equiv a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon$ and we get the action:

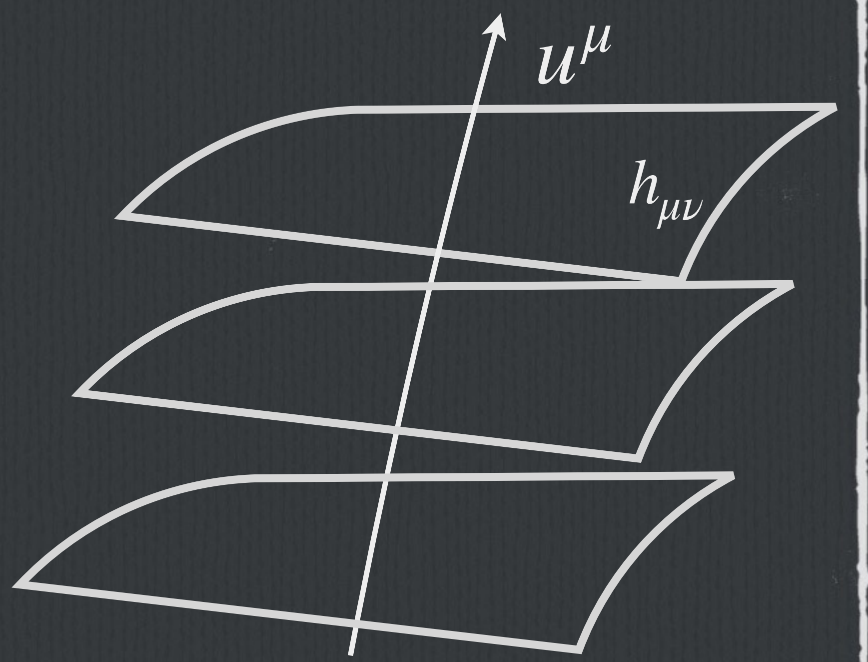
$$S = \frac{1}{2}\int d\tau\, d^3x\left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right], \quad {}' = \partial_\tau$$

Inflationary perturbations

ADM formalism

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N_i dt)(dx^j + N_j dt), \quad N \to \text{lapse}, \quad N^i \to \text{shift}.$$

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\,\Big(NR^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) +$$
$$+ N^{-1}(\dot\phi - N^i\partial_i\phi)^2 - Ng^{ij}\partial_i\phi\partial_j\phi - 2V\Big).$$



$$E_{ij} \equiv \frac{1}{2}\left(\dot g_{ij} - \nabla_i N_j - \nabla_j N_i\right), \quad E = E^i_i.$$

$$K_{ij} = N^{-1}E_{ij}.$$

Extrinsic curvature of a section

$$\nabla_i[N^{-1}(E^i_j - \delta^i_j E)] = 0,$$
$$R^{(3)} - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot\phi^2 = 0.$$

Hamiltonian constraints

Inflationary perturbations

ADM formalism

First order solution of the constraint equations

$$N_i \equiv \psi_{,i} + \tilde{N}_i\,, \quad \text{where} \quad \tilde{N}_{i,i} = 0\,, \qquad N \equiv 1 + \alpha\,.$$

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 + \dots\,, \\ \psi &= \psi_1 + \psi_2 + \dots\,, \\ \tilde{N}_i &= \tilde{N}_i^{(1)} + \tilde{N}_i^{(2)} + \dots\,, \end{aligned}$$

$$\alpha_1 = \frac{\dot{\mathcal{R}}}{H}\,, \qquad \partial^2 \tilde{N}_i^{(1)} = 0\,.$$

$$\psi_1 = -\frac{\mathcal{R}}{H} + \frac{a^2}{H}\epsilon_{\rm v}\,\partial^{-2}\dot{\mathcal{R}}\,,$$

Inflationary perturbations

PS of Scalar Perturbations.

We can go to Fourier space:

$$v(\tau, x) = \int \frac{d^3x}{(2\pi)^3} v_k(\tau) e^{i\vec{k}\cdot\vec{x}},$$

so, the e.o.m becomes:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0.$$

In de Sitter space $\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}$

Solution and PS.

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0, \Rightarrow v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)$$

The PS of the variable $\psi = a^{-1} v$ is:

$$< \psi_k \psi_{k'} >= (2\pi)^3 \delta(k + k') \frac{|v_k(\tau)^2|}{a^2} = (2\pi)^3 \delta(k + k') \frac{H^2}{2k^3}(1 + k^2\tau^2)$$

Inflationary perturbations

PS of $\mathcal{R}$.

$\mathcal{R} = \frac{H}{\dot{\phi}}\psi$ at the time of horizon crossing $a(t_*)H(t_*) = k$:

$$< \mathcal{R}_k \mathcal{R}_{k'} >= (2\pi)^3 \delta(k + k') \frac{H_*^2}{2k^3} \frac{H_*^2}{\dot{\phi}_*^2}, \quad \Delta_{\mathcal{R}}^2(k) = \frac{H_*^2}{2k^3} \frac{H_*^2}{\dot{\phi}_*^2}.$$

For slow-roll inflation:

$$\Delta_s^2(k) \approx \frac{1}{24\pi^2} \frac{V}{M_{Pl}^2} \frac{1}{\epsilon_v^*}, \quad n_s - 1 = 2\eta_v^* - 6\epsilon_v^*.$$

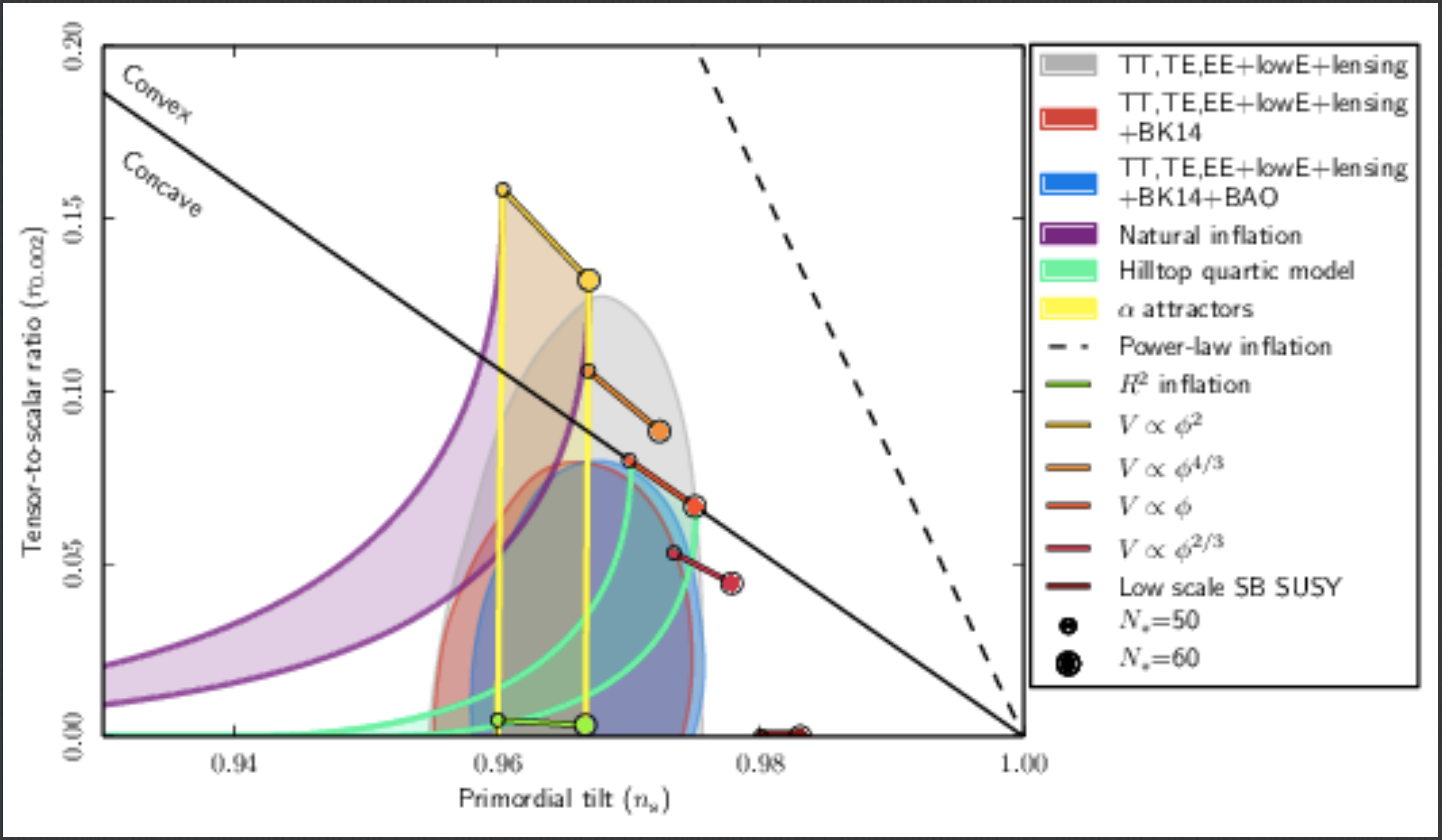
Nearly scale invariant spectrum!
We can do the same for tensor perturbations and obtain:

$$\Delta_t^2(k) \approx \frac{2}{3\pi^2} \frac{V}{M_{Pl}^2}, \quad n_t = -2\eta_v^*.$$

Additionally, $r_t \equiv \frac{\Delta_t^2(k)}{\Delta_2^2(k)} = 16\epsilon_v^* = -8n_t$.

Inflationary perturbations

Planck 2018 results for r and n_s



Final remarks

1. Inflation is a theoretical proposal that solves several problems of the unusual properties of the early universe
2. Inflationary perturbations can be calculated at linear regime for super horizon scales limit and its result gives us information about the statistical distribution of observed temperature fluctuations at the CMB.
3. There are several approaches and techniques to evaluate the evolution of cosmological perturbations.
4. Cosmological perturbation theory is also used at different scales from those involved in inflation, for instance at large scales during cold dark matter dominated epoch.