Recent Advances in Percolation Theory

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Tutorial Course

4th Workshop on Statistical Physics

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Content of the course

Basic notions of percolation

Fractal subsets at criticality

Variants of percolation

Percolation on correlated surfaces

Schramm-Loewner Evolution

Explosive percolation models

Breakdown models

History

Broadbent and Hammersley
Proc. Cambridge Phil. Soc.
Vol. 53, p.629 (1957)

John M. Hammersley
(1920 – 2004)

References to percolation

- **D. Stauffer: „Introduction to Percolation Theory"** (Taylor and Francis, 1985)
- **D. Stauffer and A. Aharony: „Introduction to Percolation Theory,** Revised Second Edition" (Taylor and Francis, 1992)
- **M. Sahimi: „Applications of Percolation Theory"** (Taylor and Francis, 1994)
- **G. Grimmett: „Percolation"** (Springer, 1989)
- **B.Bollobas and O.Riordan: „Percolation"** (Cambridge Univ. Press, 2006)

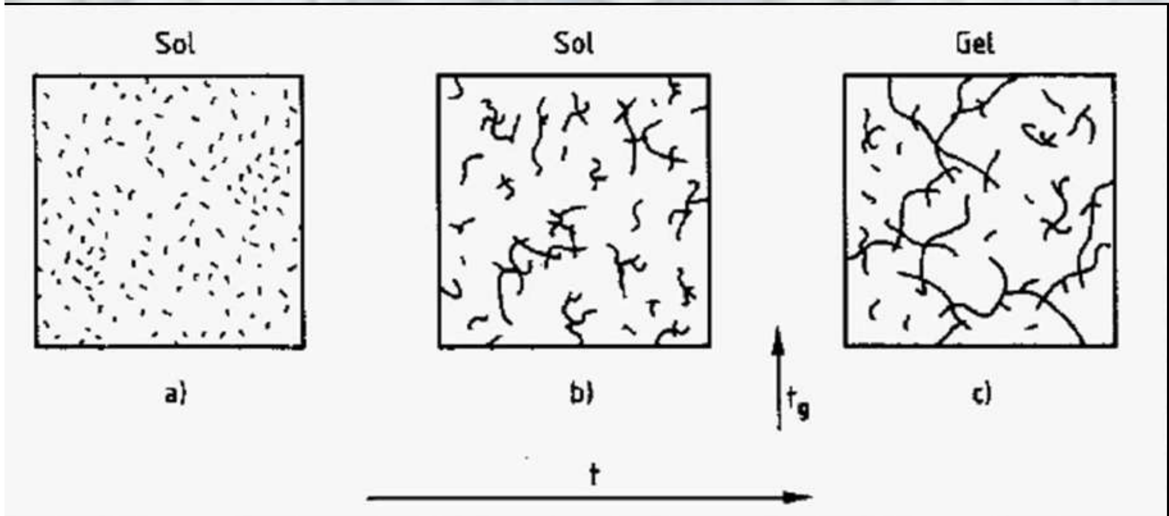
Percolator

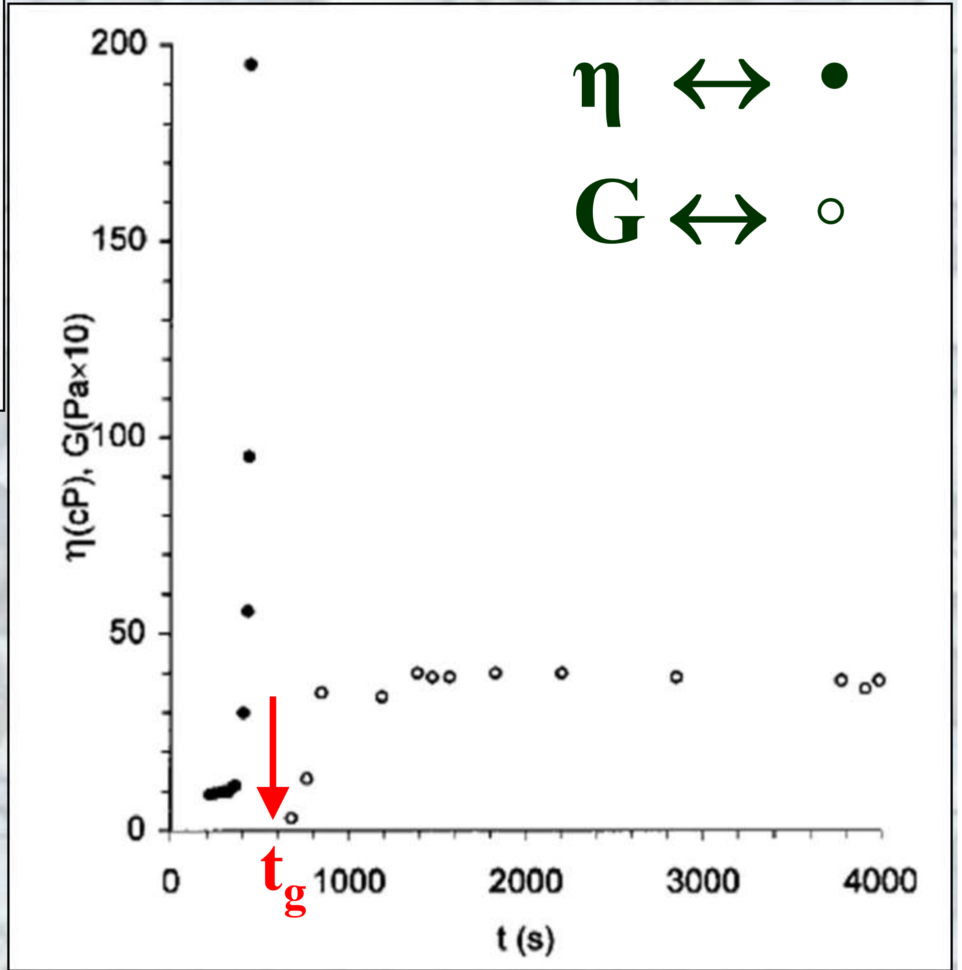
Applications of percolation

- Porous media (oil production, pollution of soils)
- Sol-gel transition
- Mixtures of conductors and insulators (or superconductors and conductors)
- Forest fires
- Propagation of epidemics or computer virus
- Crash of stock markets (Sornette)
- Landslide election victories (Galam)
- Recognition of antigens by T-cells (Perelson)
- …

Gelatin formation

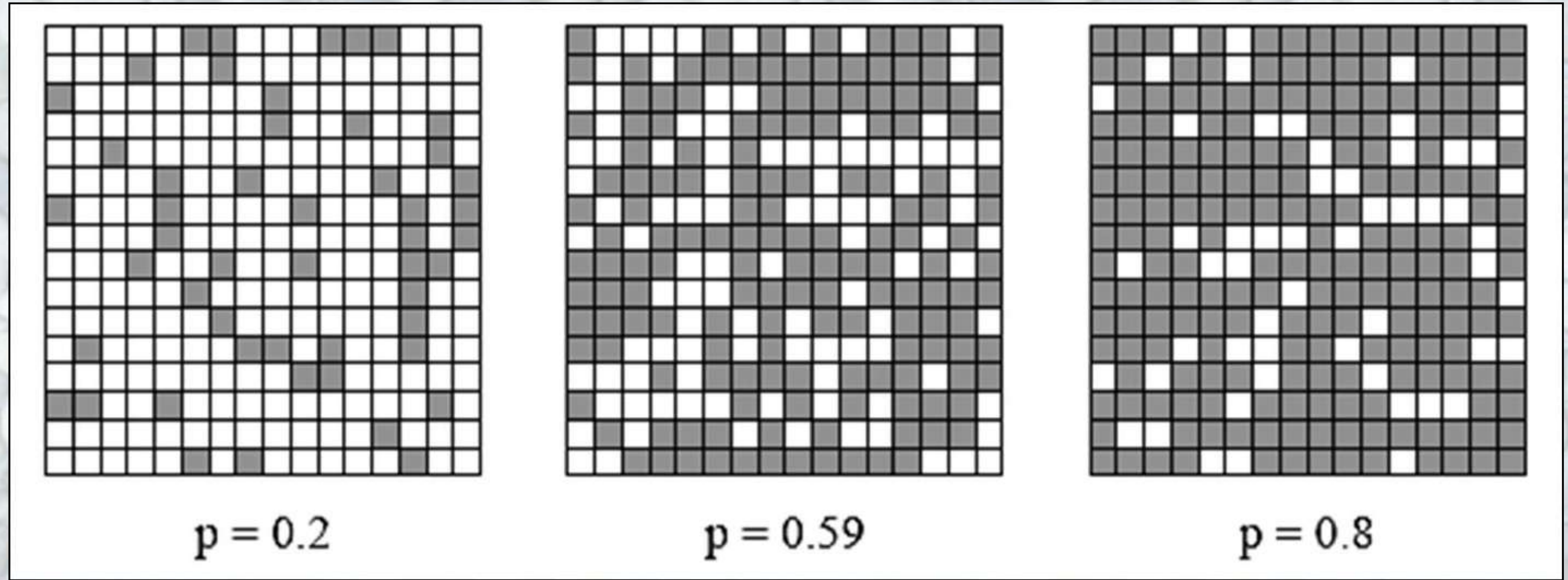
Sol -gel transition





Shear modulus G vanishes and viscosity η diverges at t_g as function of time t.

Percolation



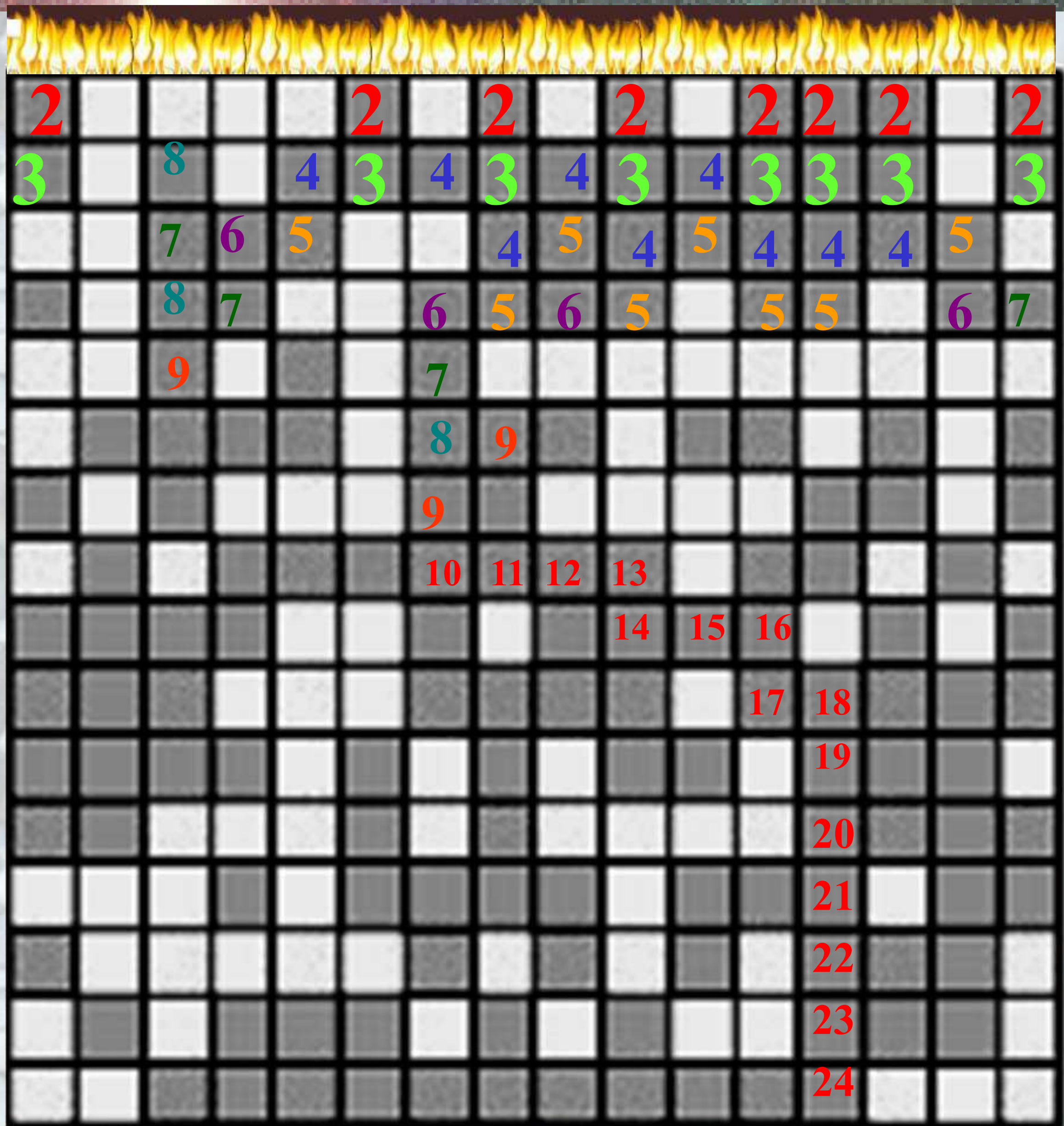
site percolation on square lattice

p is the probability to occupy a site.

Neighboring occupied sites are „connected“

and belong to the same cluster.

Burning method



$p = 0.59$

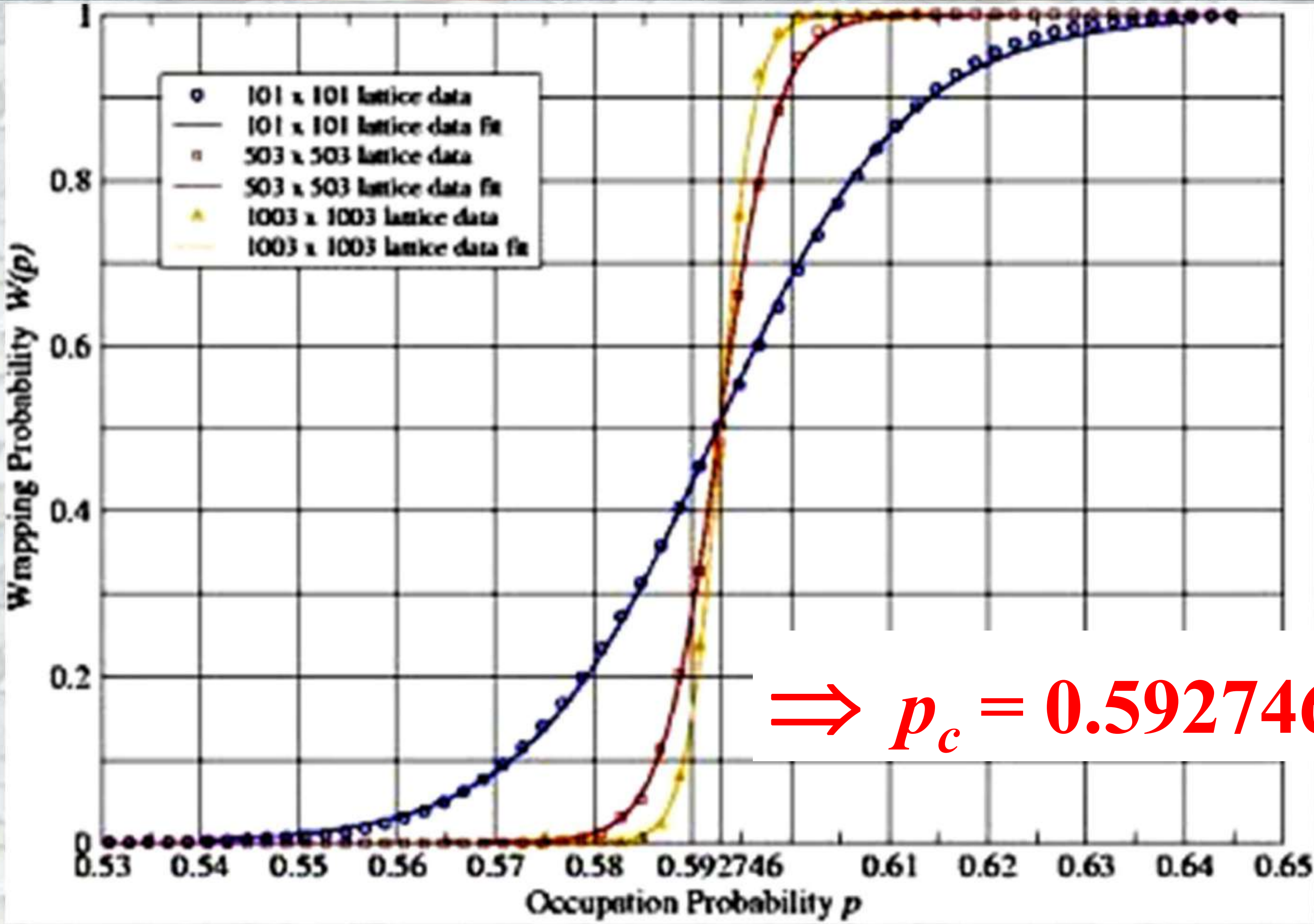
$L = 16$

shortest path

$t_s = 24$

HH et al (1984)

Probability to find a spanning cluster



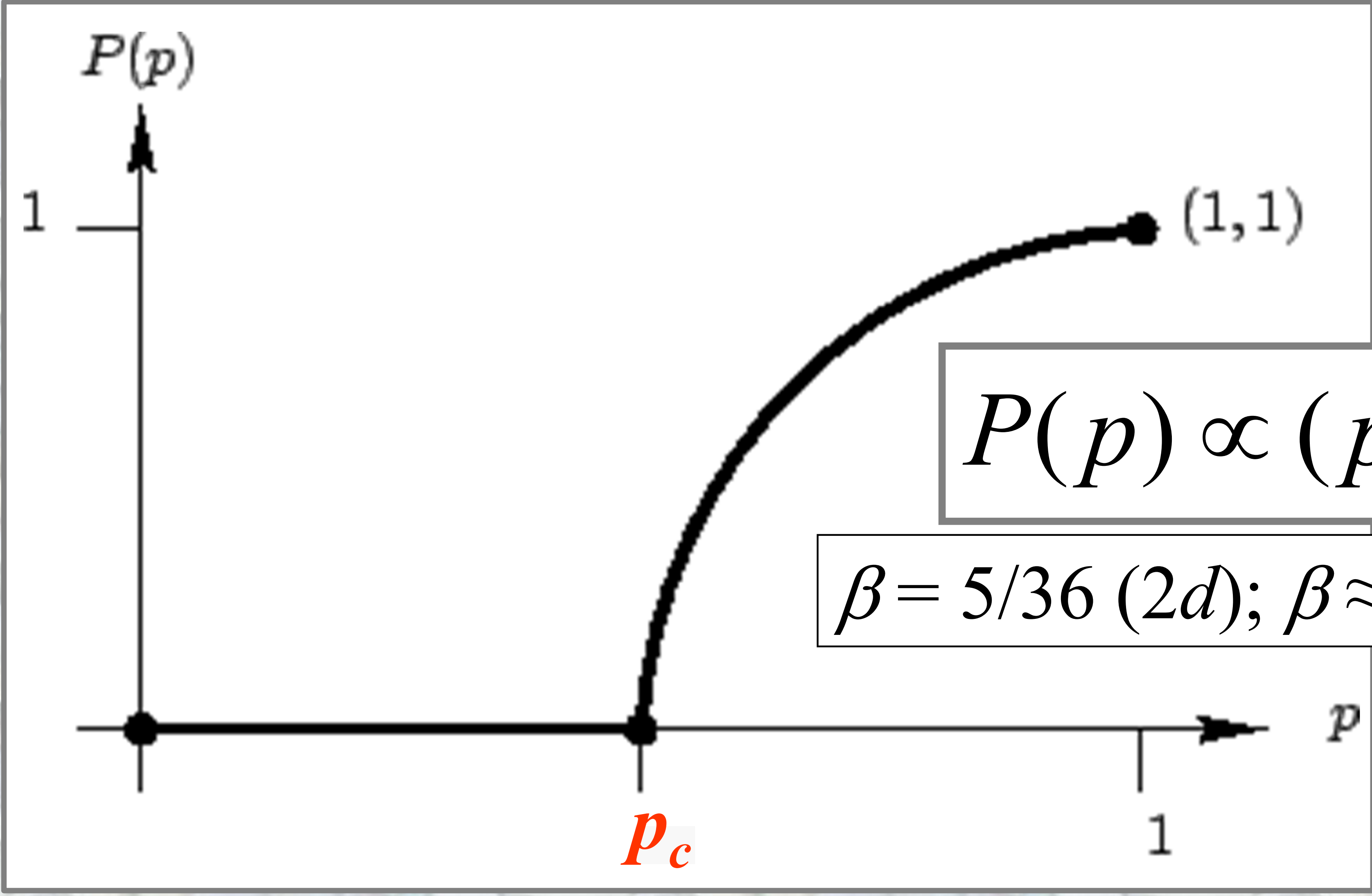
$\Rightarrow p_c = 0.592746...$

Percolation thresholds p_c

lattice	site	bond
cubic (body-centered)	0.246	0.1803
cubic (face-centered)	0.198	0.119
cubic (simple)	0.3116	0.2488
diamond	0.43	0.388
honeycomb	0.6962	0.65271*
4-hypercubic	0.197	0.1601
5-hypercubic	0.141	0.1182
6-hypercubic	0.107	0.0942
7-hypercubic	0.089	0.0787
square	0.592746	0.50000*
triangular	0.50000*	0.34729*

Order parameter of percolation

$P(p)$ = fraction of sites in the largest cluster



p_c

$$P(p) \propto (p - p_c)^{\beta}$$

$\beta = 5/36$ (2d); $\beta \approx 0.41$ (3d)

Many clusters

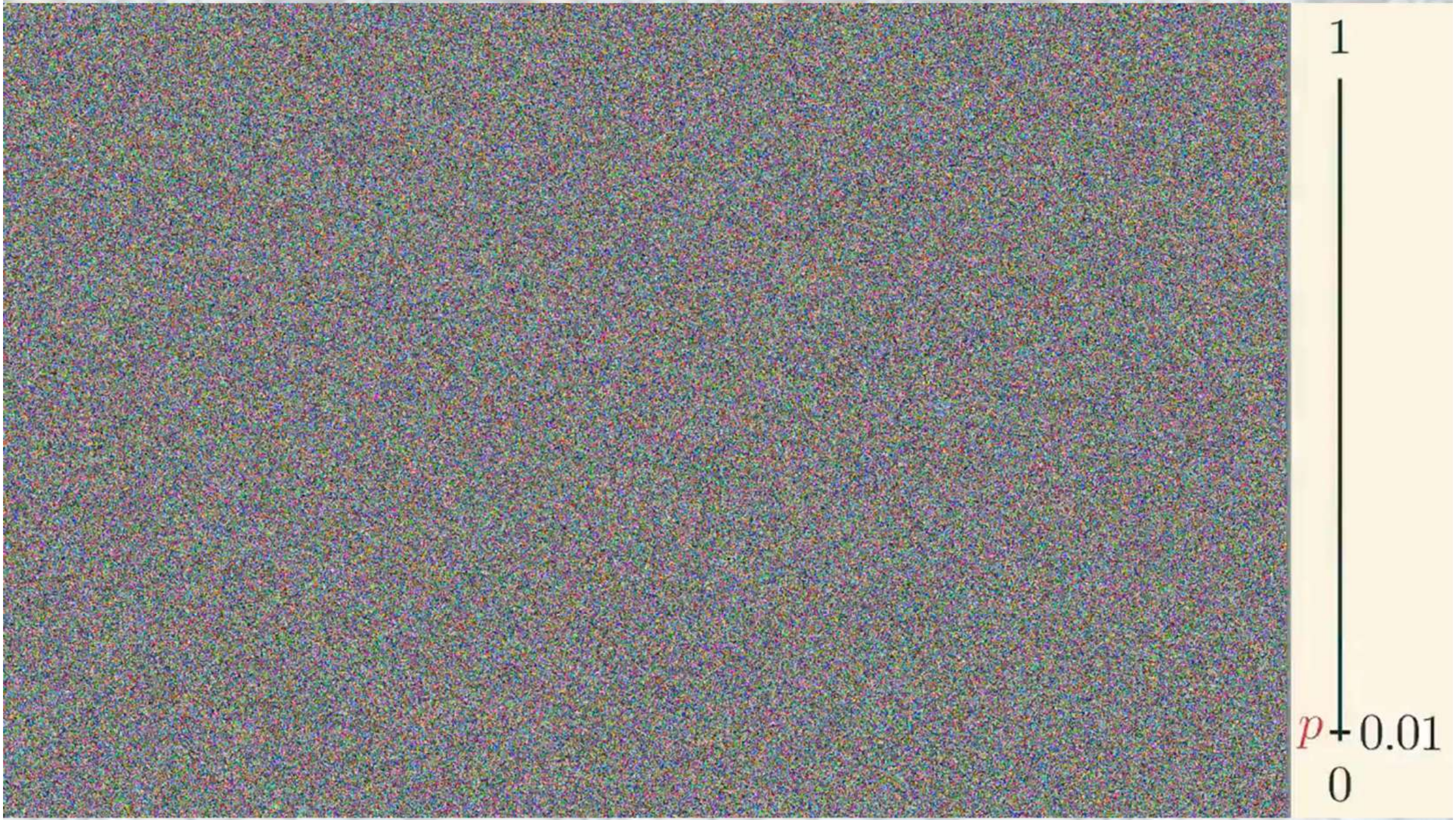
bond

percolation

We have clusters of different sizes s and can study the cluster size distribution n_s

$$n_s = \frac{N_s}{N}$$

Many clusters



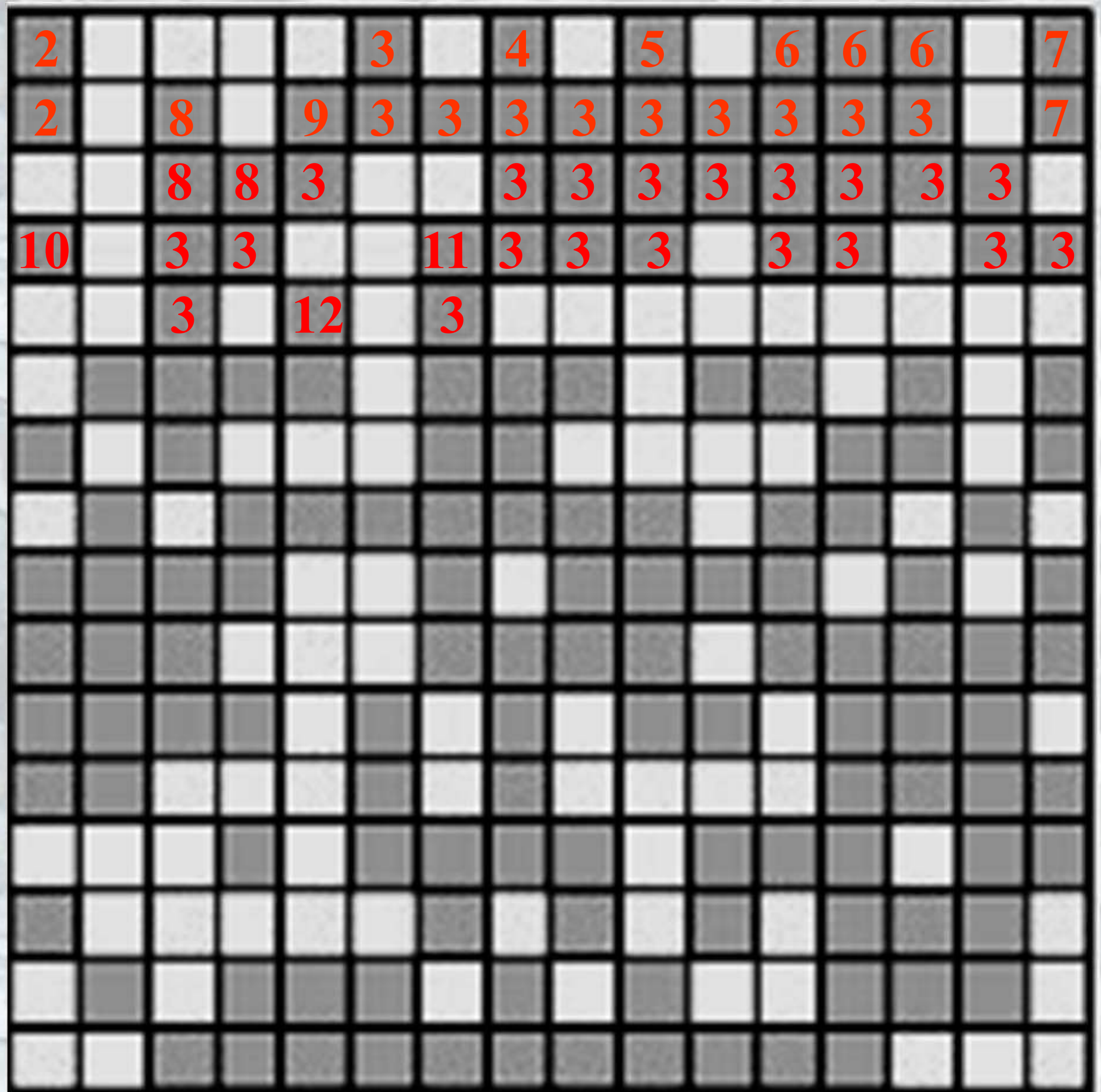
Cluster size distribution

Hoshen-Kopelman Algorithm (1976)

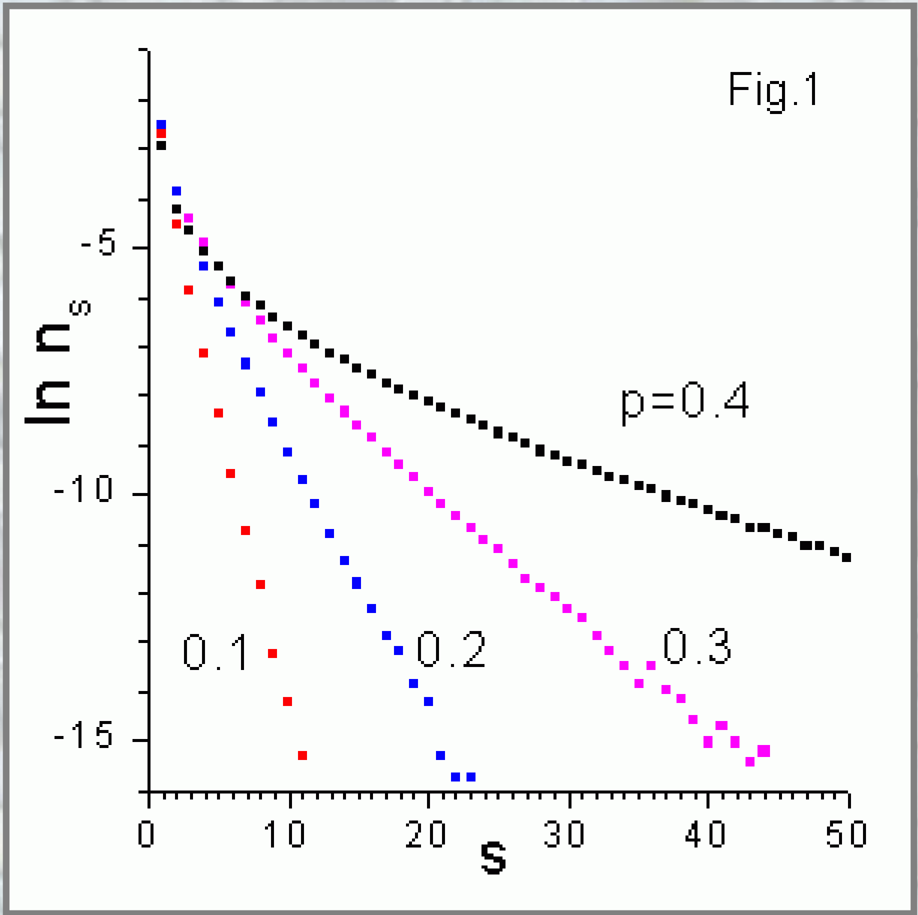

Raoul Kopelman

- $N(i,j) \in \{0,1\}$, 0 = empty, 1 = occupied
- Start: $k = 2$, N(first occupied site) = k, $M(k) = 1$
- If site top and left are empty: $k = k + 1$ and continue
- If one of them has value k_0: $N(i,j) = k_0$, $M(k_0) = M(k_0) + 1$
- If both are occupied with k_1 and k_2: choose one, e.g. k_1 , $N(i,j) = k_1$, $M(k_1) = M(k_1) + M(k_2) + 1$, $M(k_2) = - k_1$
- If any k has negative M(k): `while(M(k)<0)k=-M(k)`
- At end: `for(k=2; k<=kmax; k++) n(M(k))=n(M(k))+1`

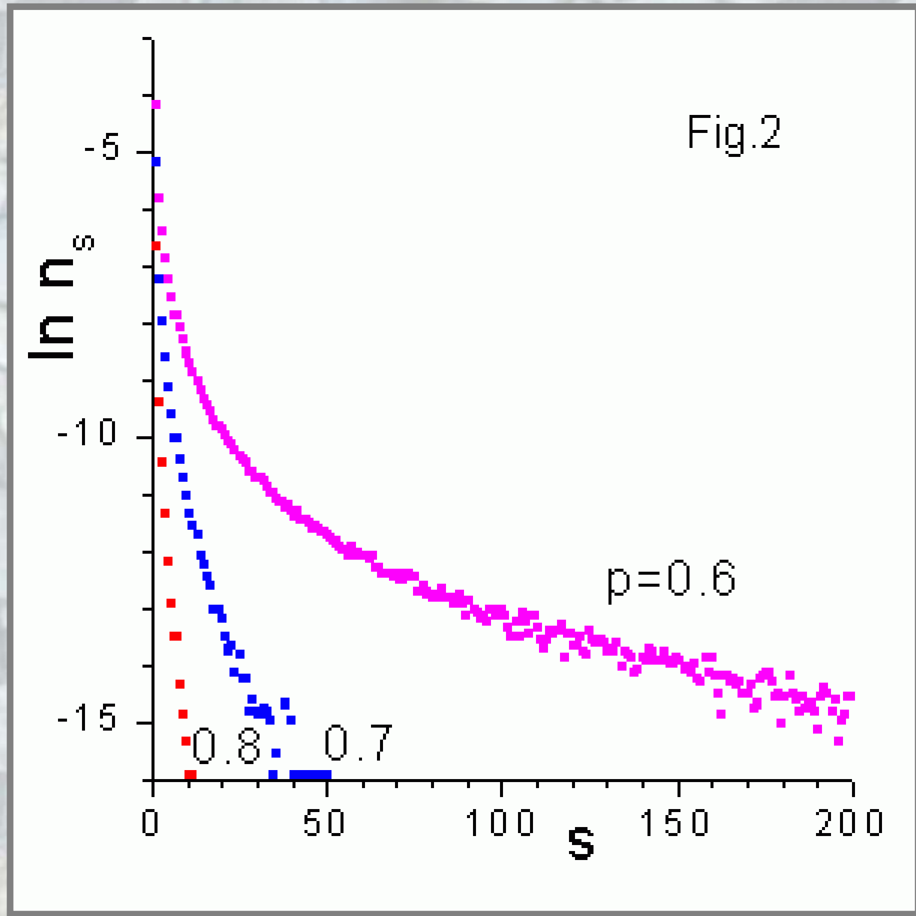
Evolution of N(i,j)



Cluster size distribution n_s

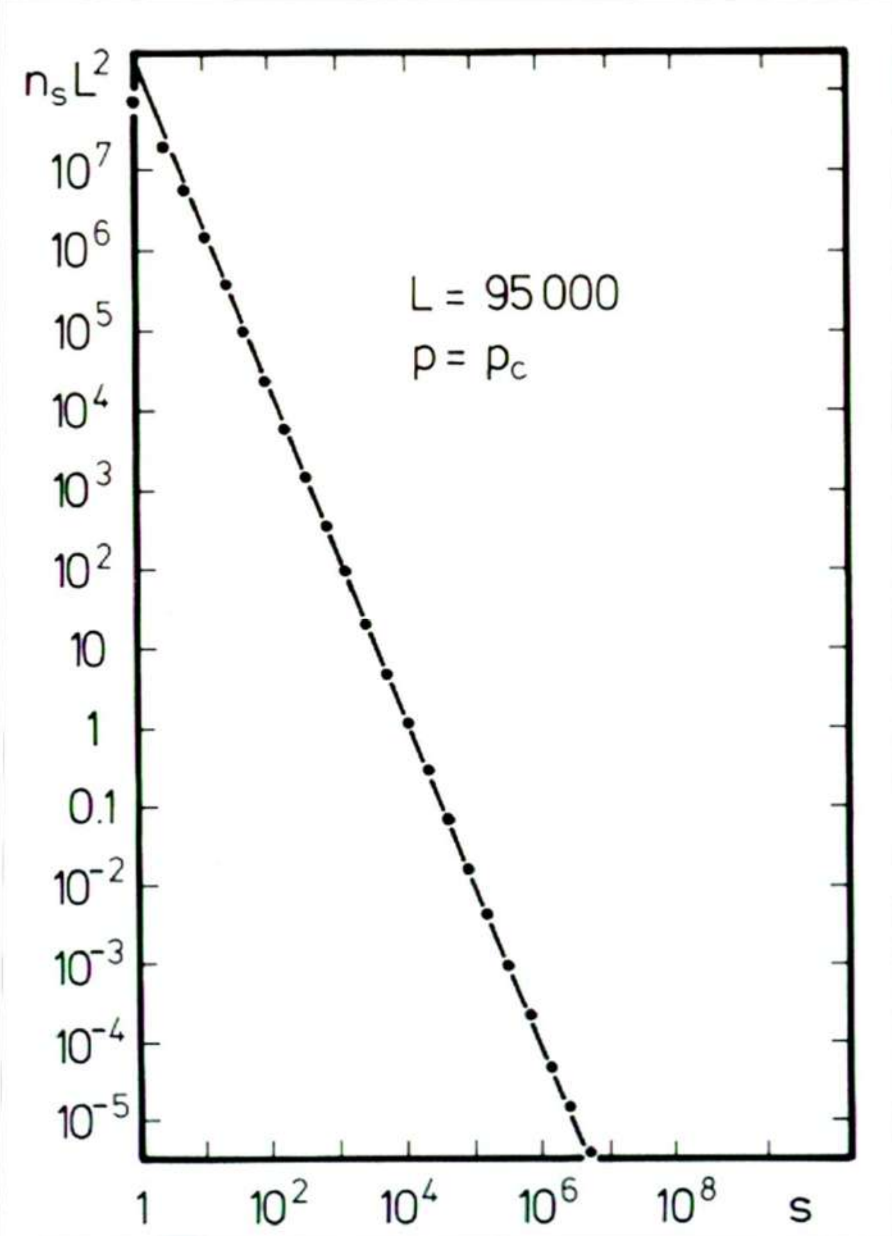


$$n_s(p < p_c) \propto s^{-\theta} e^{-as}$$



$$n_s(p > p_c) \propto e^{-bs^{(1-1/d)}}$$

Cluster size distribution at p_c



at p_c

$$n_s \propto s^{-\tau}$$

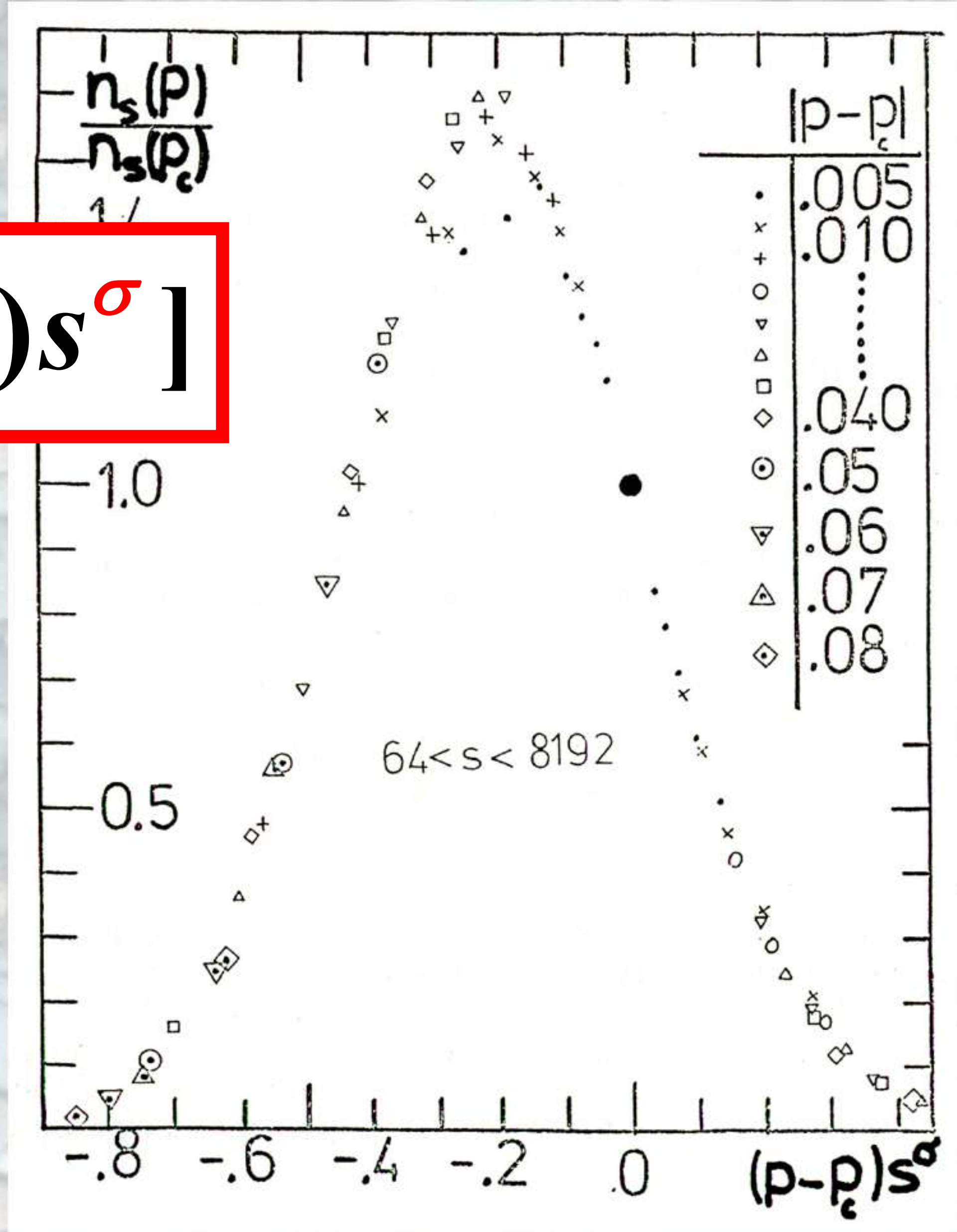
$$\tau = \begin{cases} \frac{187}{91} & \text{in} \quad 2d \\ 2.18 & \text{in} \quad 3d \end{cases}$$

$$2 \le \tau \le \frac{5}{2}$$

Scaling of cluster size distribution

s = size of cluster

$$n_s(p) = s^{-\tau}\,\Re_{\pm}[(p-p_c)s^{\sigma}]$$



Second moment χ

$$\chi = \langle \sum_s{}' s^2 n_s \rangle$$

$'$ means that one excludes the largest cluster

$$\chi \propto C_{\pm} |p - p_c|^{-\gamma}$$

$\gamma = 43/18 \approx 2.39$ (2d)

$\gamma \approx 1.80$ (3d)

$$\gamma = \frac{3 - \tau}{\sigma}$$

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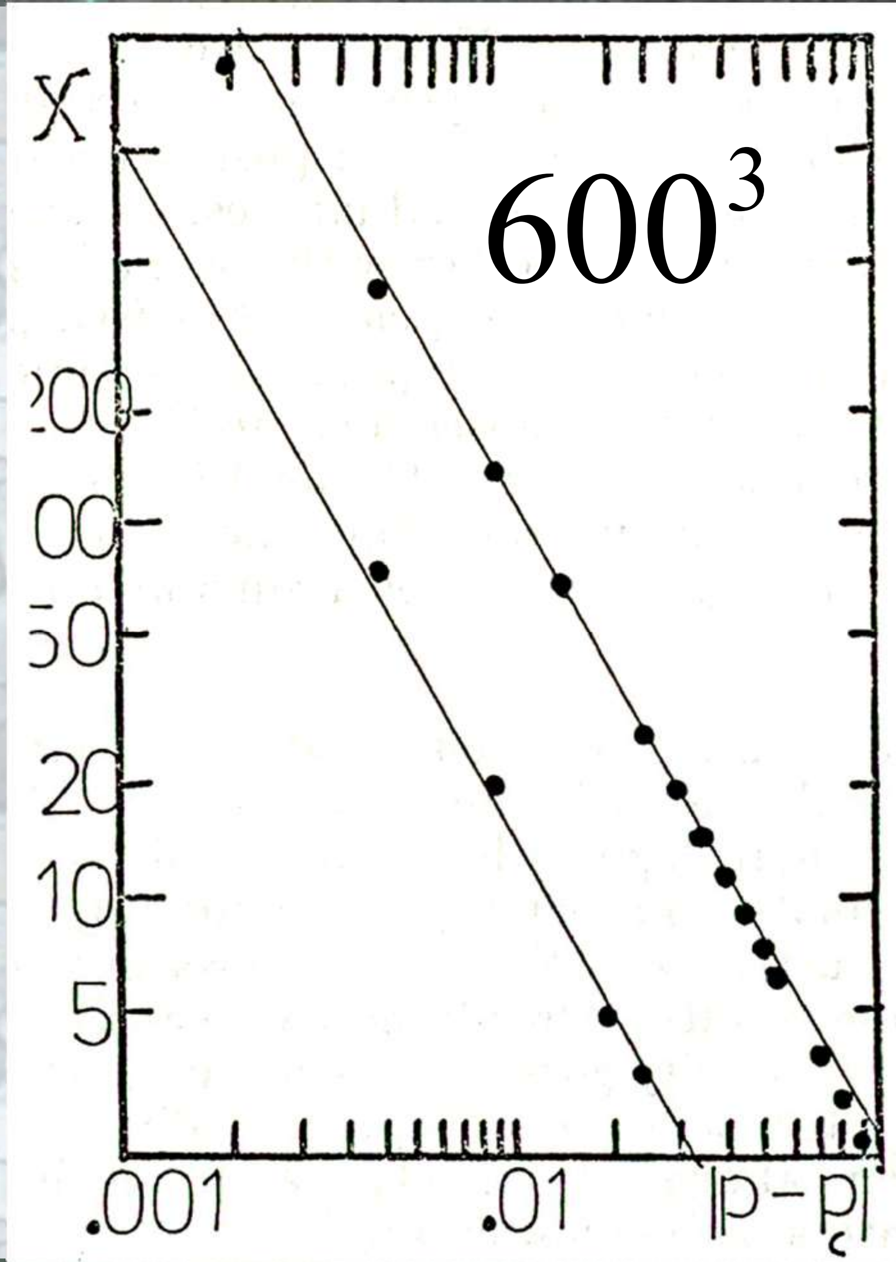


Critical exponents

Table 2. Percolation exponents for $d = 2, 3, 4, 5, 6 - \varepsilon$ and in the Bethe lattice together with the page number defining the exponent. Rational numbers give (presumably) exact results, whereas those with a decimal fraction are numerical estimates.

Exponent	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6 - \varepsilon$	Bethe	Page
α	−2/3	−0·62	−0·72	−0·86	$-1 + \varepsilon/7$	−1	39
β	5/36	0·41	0·64	0·84	$1 - \varepsilon/7$	1	37
γ	43/18	1·80	1·44	1·18	$1 + \varepsilon/7$	1	37
ν	4/3	0·88	0·68	0·57	$\frac{1}{2} + 5\varepsilon/84$	1/2	60
σ	36/91	0·45	0·48	0·49	$\frac{1}{2} + O(\varepsilon^2)$	1/2	35
τ	187/91	2·18	2·31	2·41	$\frac{5}{2} - 3\varepsilon/14$	5/2	33
$D(p = p_c)$	91/48	2·53	3·06	3·54	$4 - 10\varepsilon/21$	4	10
$D(p < p_c)$	1·56	2	12/5	2·8	–	4	62
$D(p > p_c)$	2	3	4	5	–	4	62
$\zeta(p < p_c)$	1	1	1	1	–	1	56
$\zeta(p > p_c)$	1/2	2/3	3/4	4/5	–	1	56
$\theta(p < p_c)$	1	3/2	1·9	2·2	–	5/2	54
$\theta(p > p_c)$	5/4	−1/9	1/8	−449/450	–	5/2	54
f_{max}	5·0	1·6	1·4	1·1		1	42
μ	1·30	2·0	2·4	2·7	$3 - 5\varepsilon/21$	3	91
s	1·30	0·73	0·4	$0{\cdot}1_5$		0	93
D_B	1·6	$1{\cdot}7_4$	1·9	2·0	$2 + \varepsilon/21$	2	95
$D_{min}(p = p_c)$	1·13	1·34	1·5	1·8	$2 - \varepsilon/6$	2	97
$D_{min}(p < p_c)$	1·17	1·36	1·5		–	2	98
$D_{max}(p = p_c)$	1·4	1·6	1·7	1·9	$2 - \varepsilon/42$	2	97

For the exponents at p_c, the Bethe lattice values are exact at $d \geqslant 6$. A dash means that 6 is not the upper critical dimension for the ε-expansion.

Size dependence of OP

***L* is linear size of the system**

at p_c:

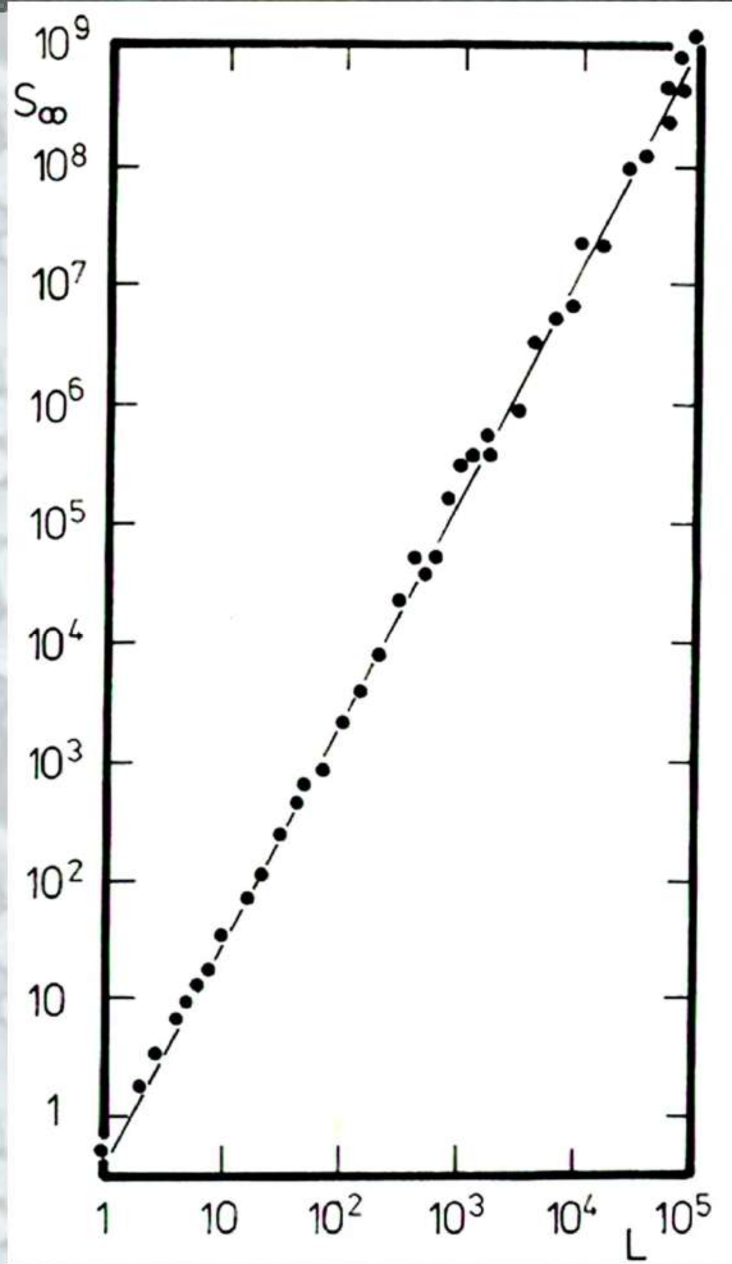
$$PL^d = s_\infty \propto L^{d_f}$$

$d_f = 91/48$ in $d = 2$

$d_f \approx 2.51$ in $d = 3$

We will show later:

$$d_f = d - \frac{\beta}{\nu}$$



Shortest path t_s at p_c

also called

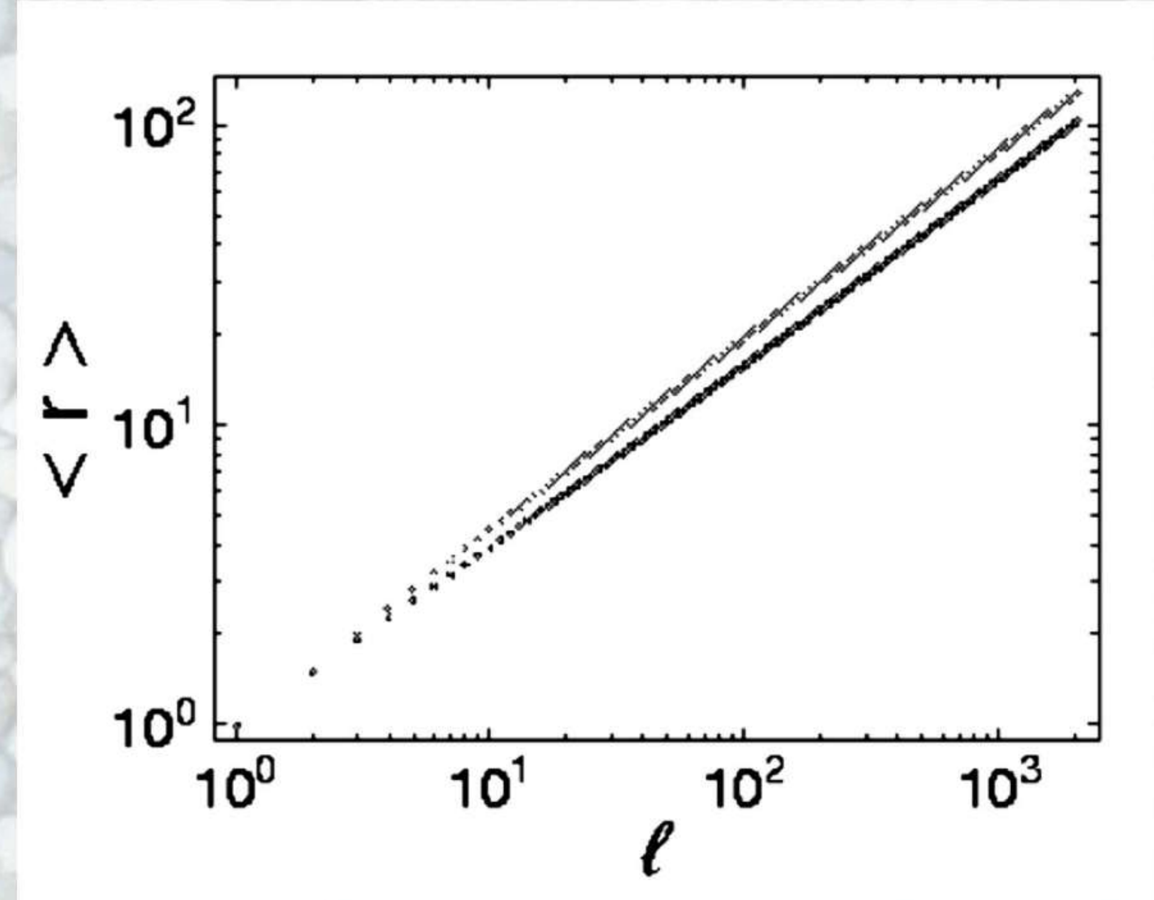
„chemical distance“ ℓ

$$t_s \propto L^{d_{\min}}$$

$d_{min} \approx 1.13$ (2d)

$d_{min} \approx 1.37$ (3d)

$d_{min} \approx 1.61$ (4d)



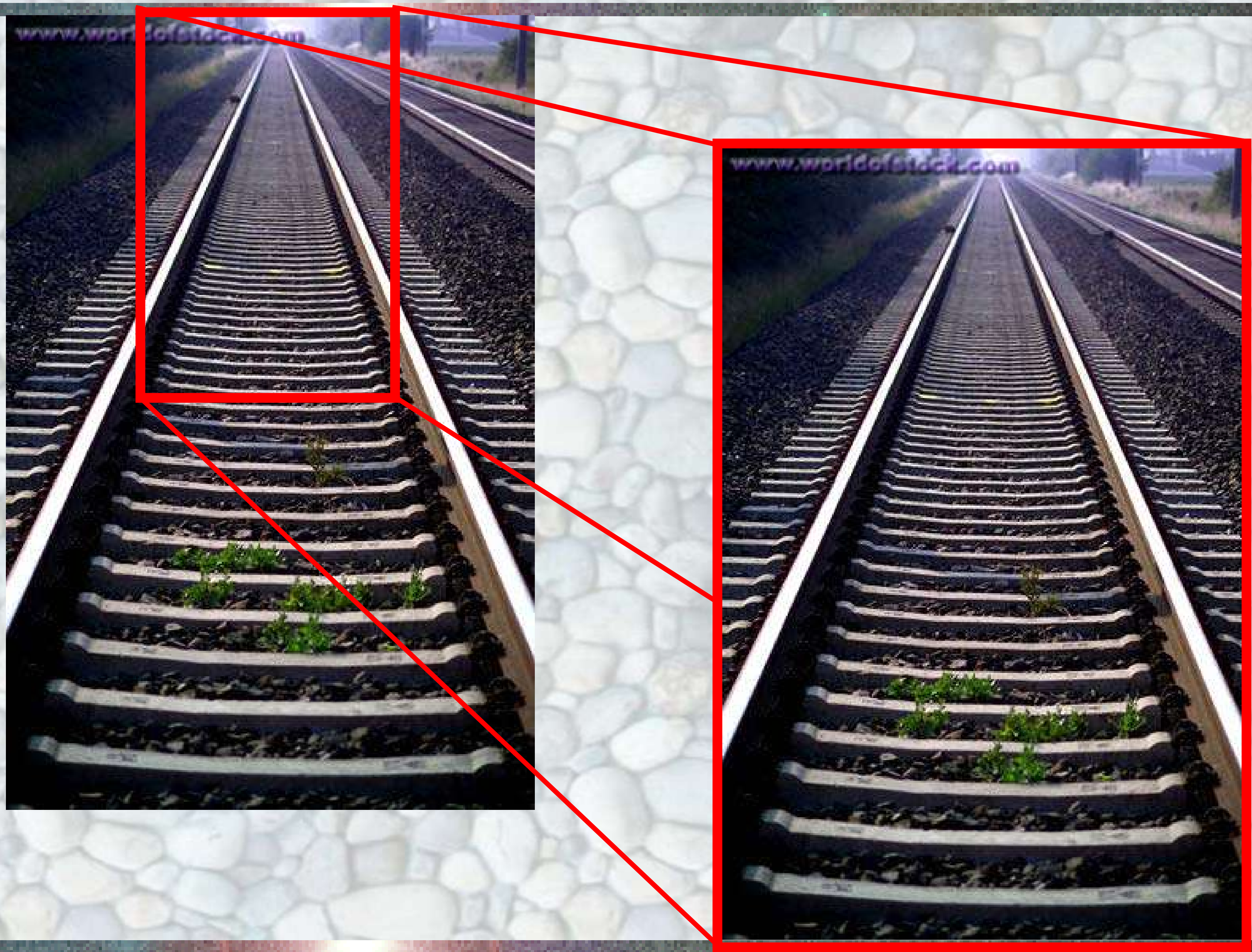
site (upper) and bond (lower) percolation in 4 dimensions (Ziff, 2001)

Fractal dimension

Books:

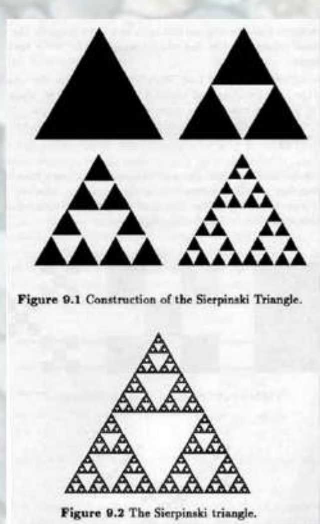
- **B.B.Mandelbrot, „Les Objets Fractals: Forme Hazard et Dimension“ (Flammarion, Paris,1975)**
- **J. Feder, „Fractals“ (Plenum Press, NY, 1988)**
- **T. Vicsek, „Fractal Growth Phenomena“ (World Scientific, Singapore, 1989)**
- **H.-O.Peitgen and P.H.Richter, „The Beauty of Fractals“ (Springer, Berlin, 1986)**
- **J.-F. Gouyet, „Physique et Structures Fractales) (Masson, Paris, 1992)**

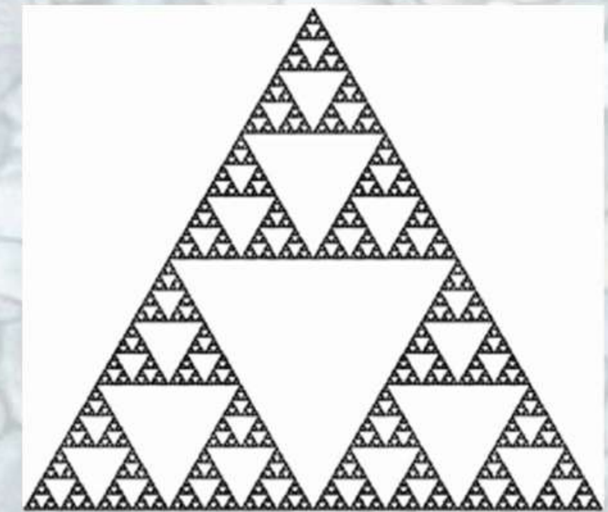
Self similarity



Fractal dimension

Sierpinski gasket

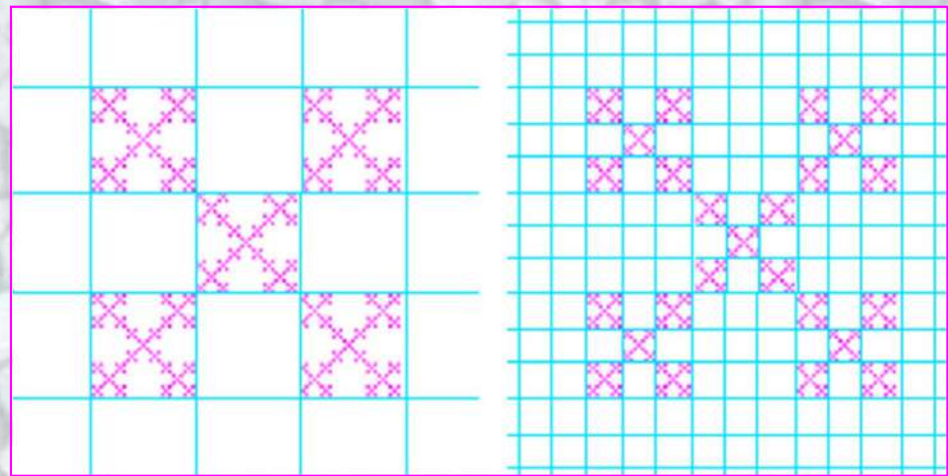




$$M \propto L^{d_f}$$

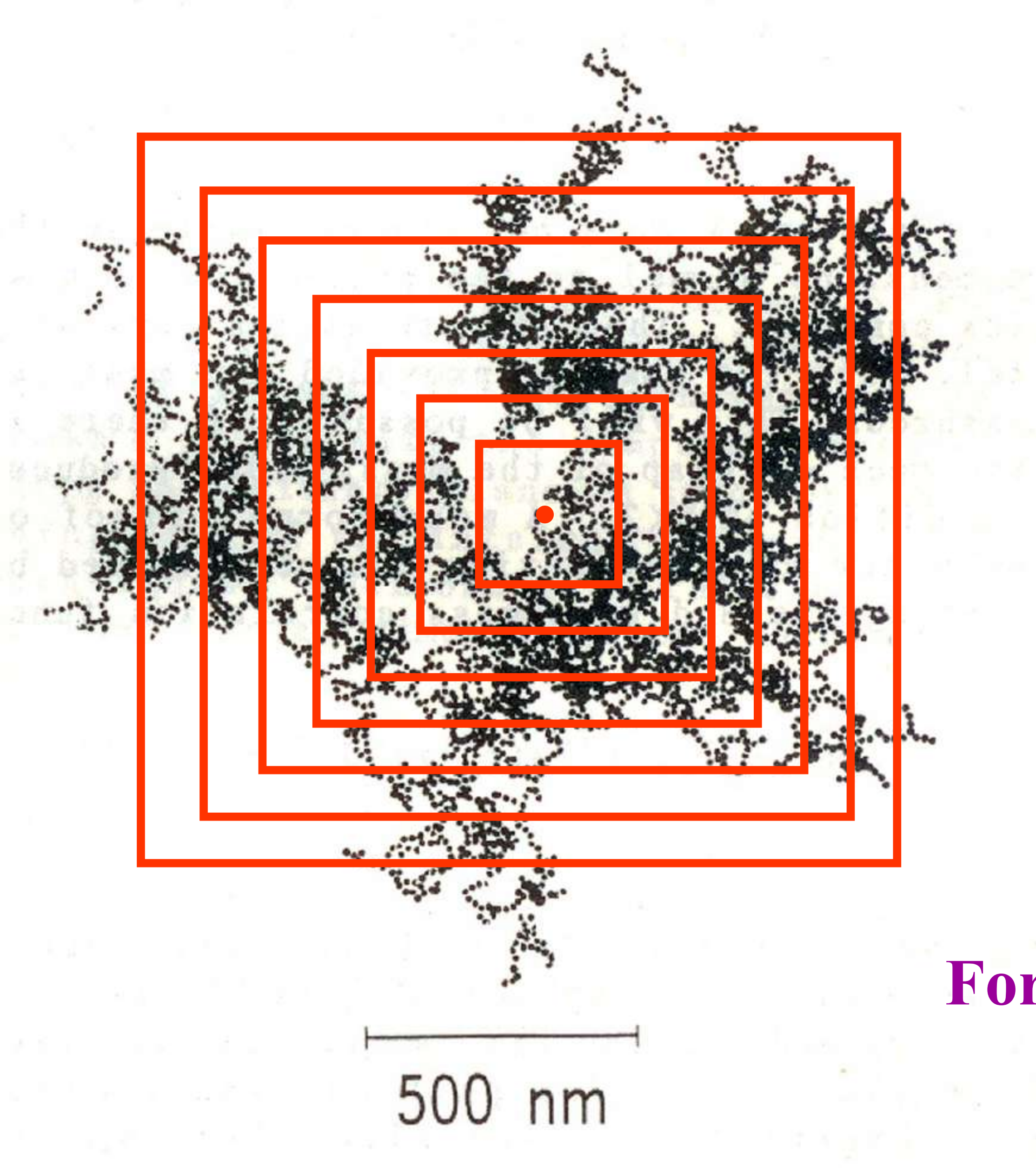
$$d_f = \log(3)/\log(2) \approx 1.602$$

„box counting“ method:



$$d_f = \log(5)/\log(3) \approx 1.46$$

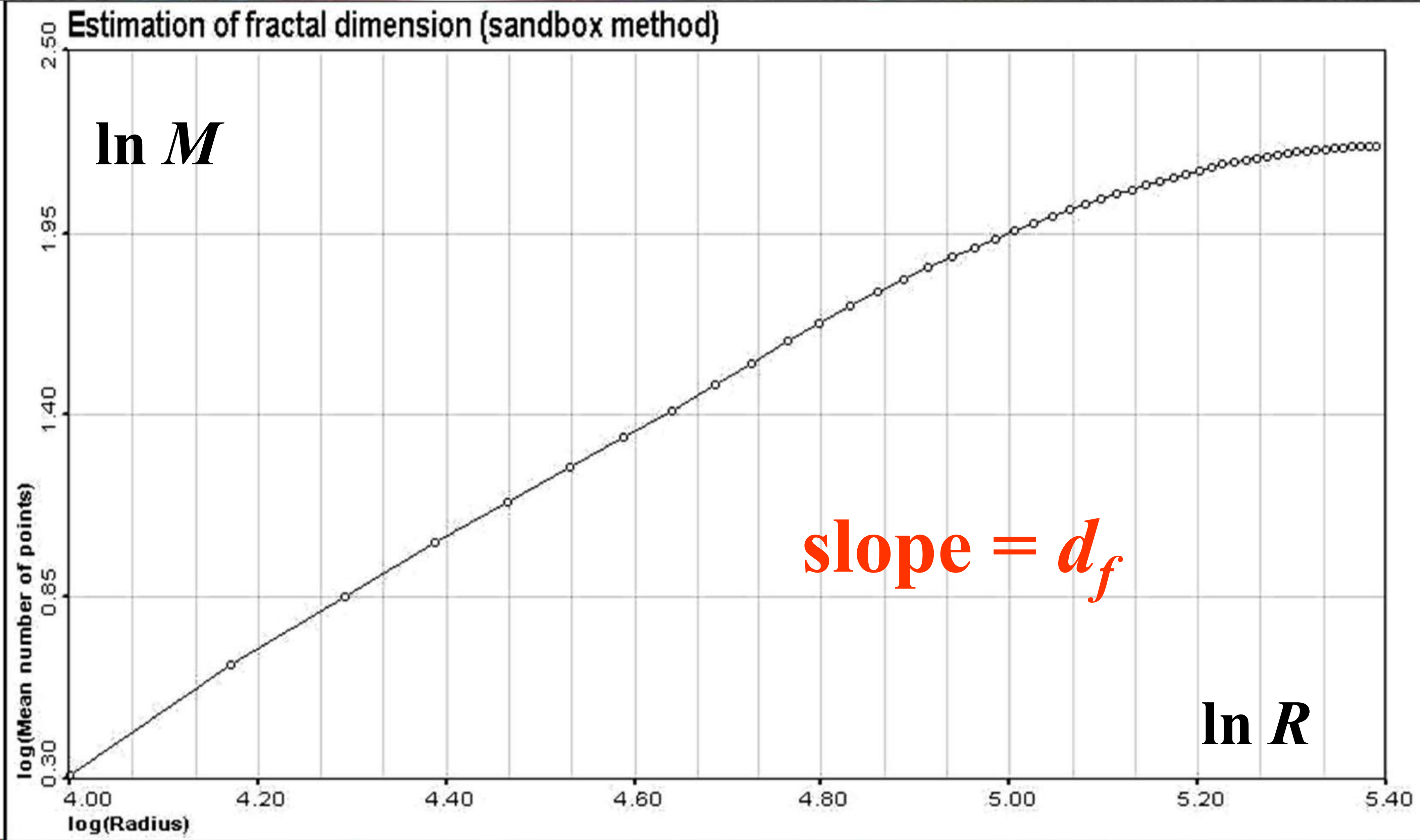
Sand-box method



M(*R*) is the number of particles in box of size *R* .

Forrest and Witten (1979)

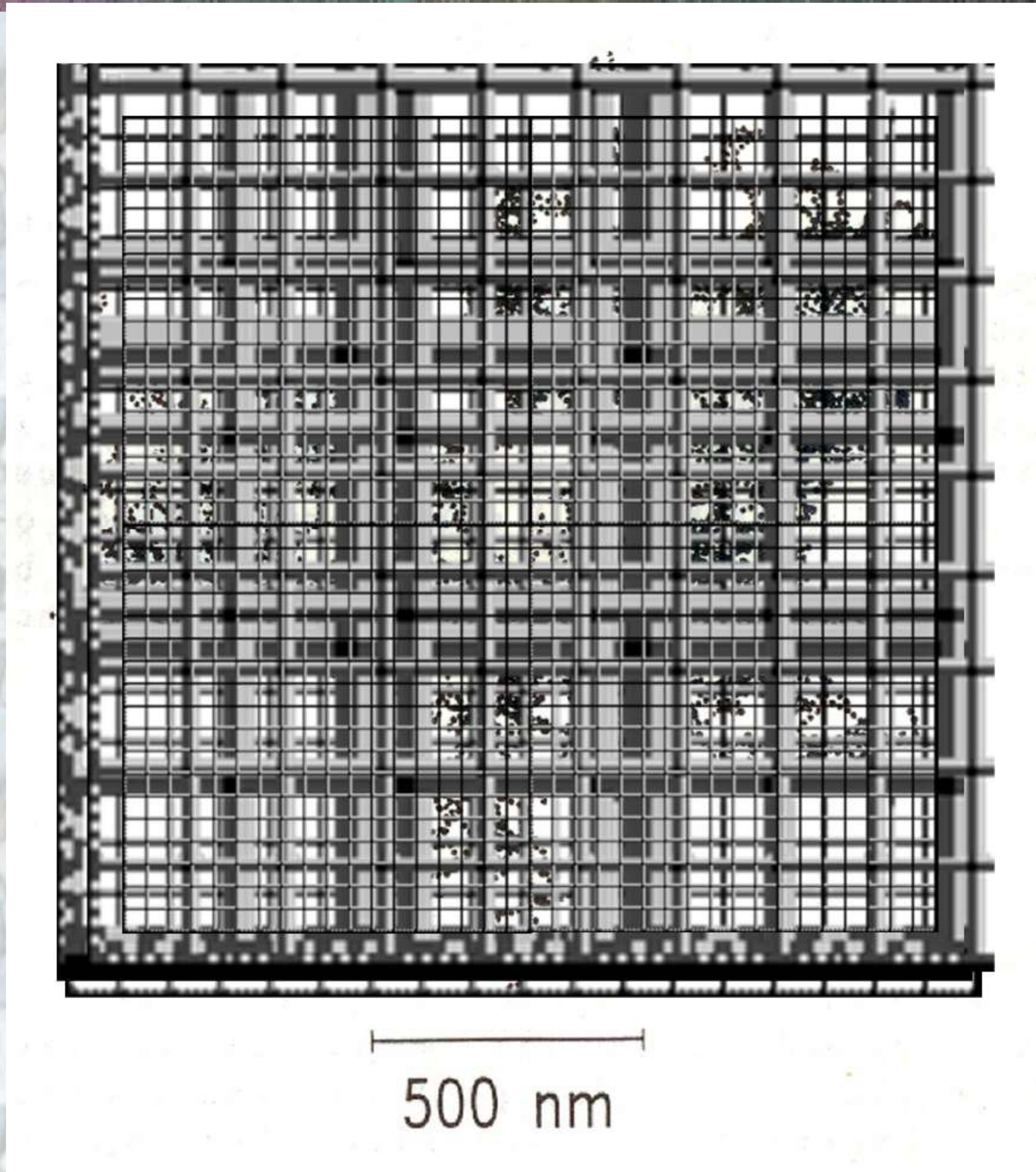
Sand-box method



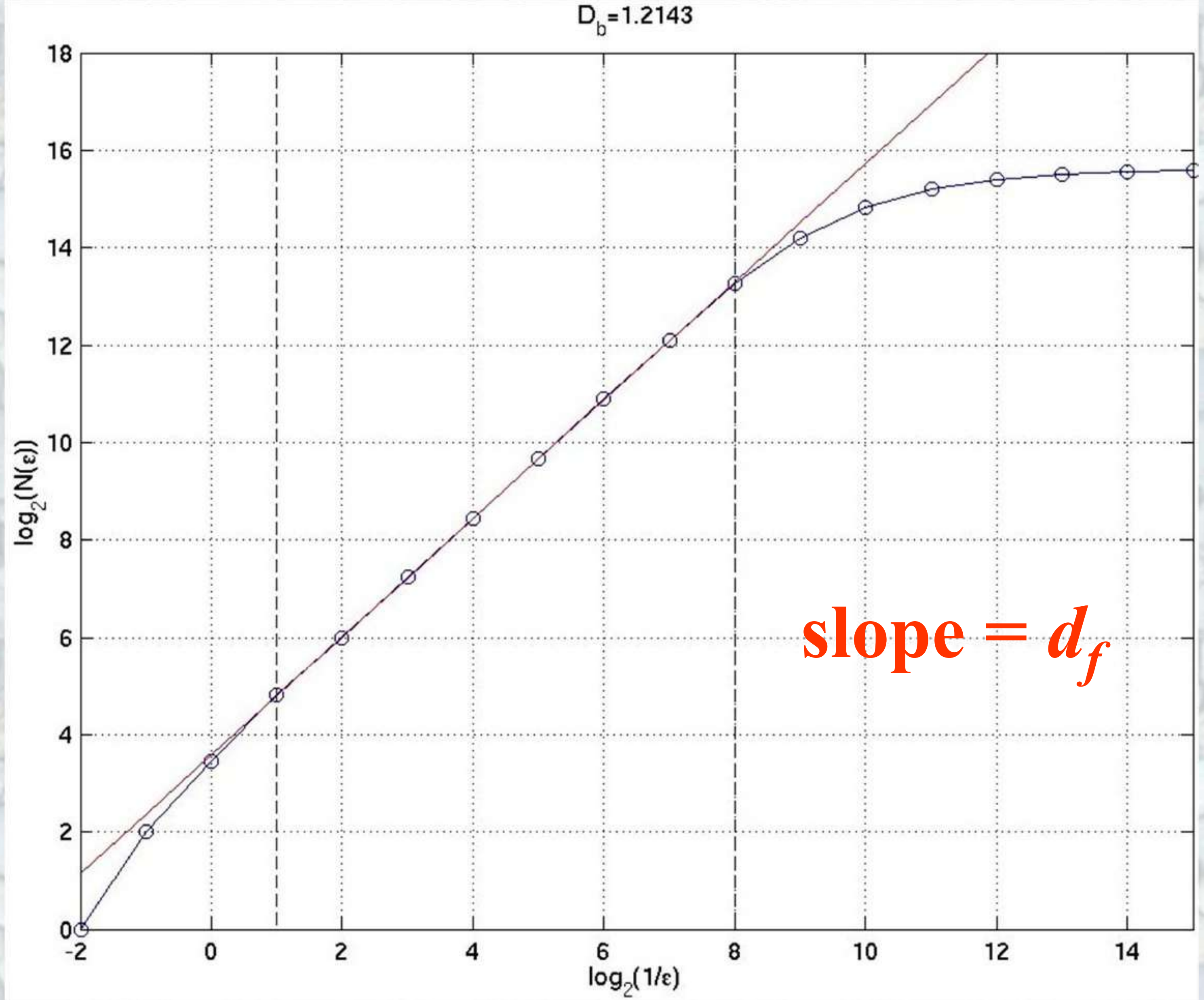
Box-counting method

ε = grid spacing

N(ε) = number of occupied cells



Box-counting method



slope = d_f

Multifractality

N_i = number of points in box i

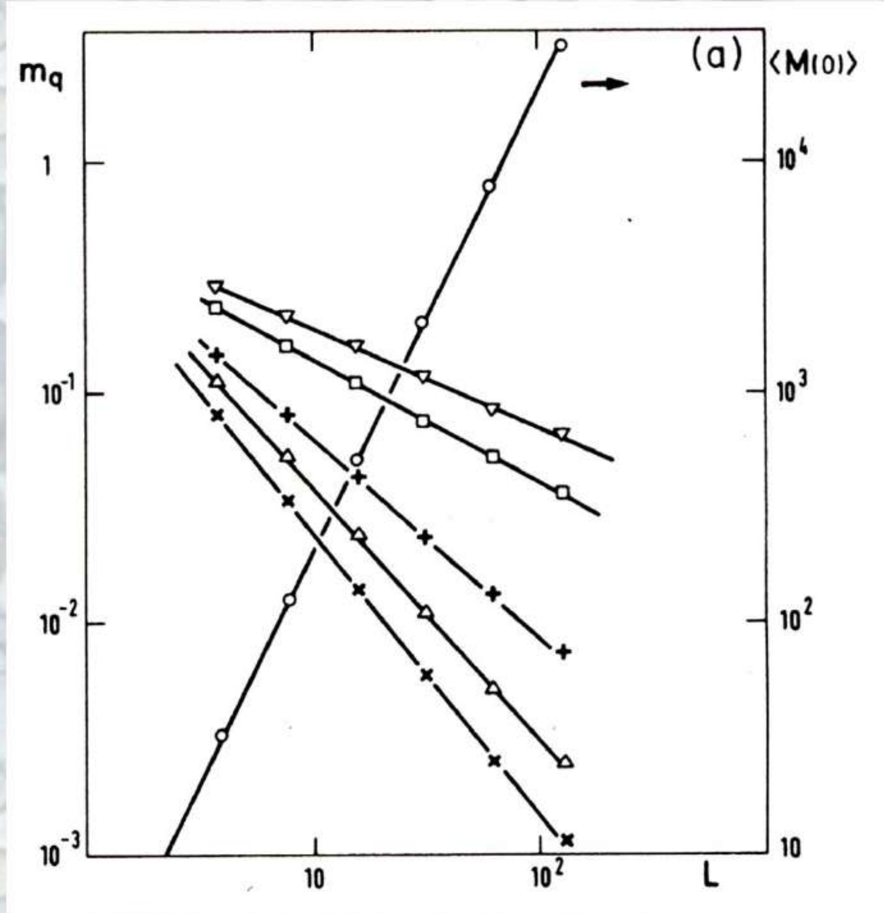
$p_i = N_i$ / total number of points

$$M_q = \sum_i p_i^q$$

$$M_q \propto L^{d_q}$$

$$m_q = (M_q / M_0)^{1/q}$$

$$d_q = \frac{1}{q-1} \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{\ln m_q}{\ln \varepsilon}$$

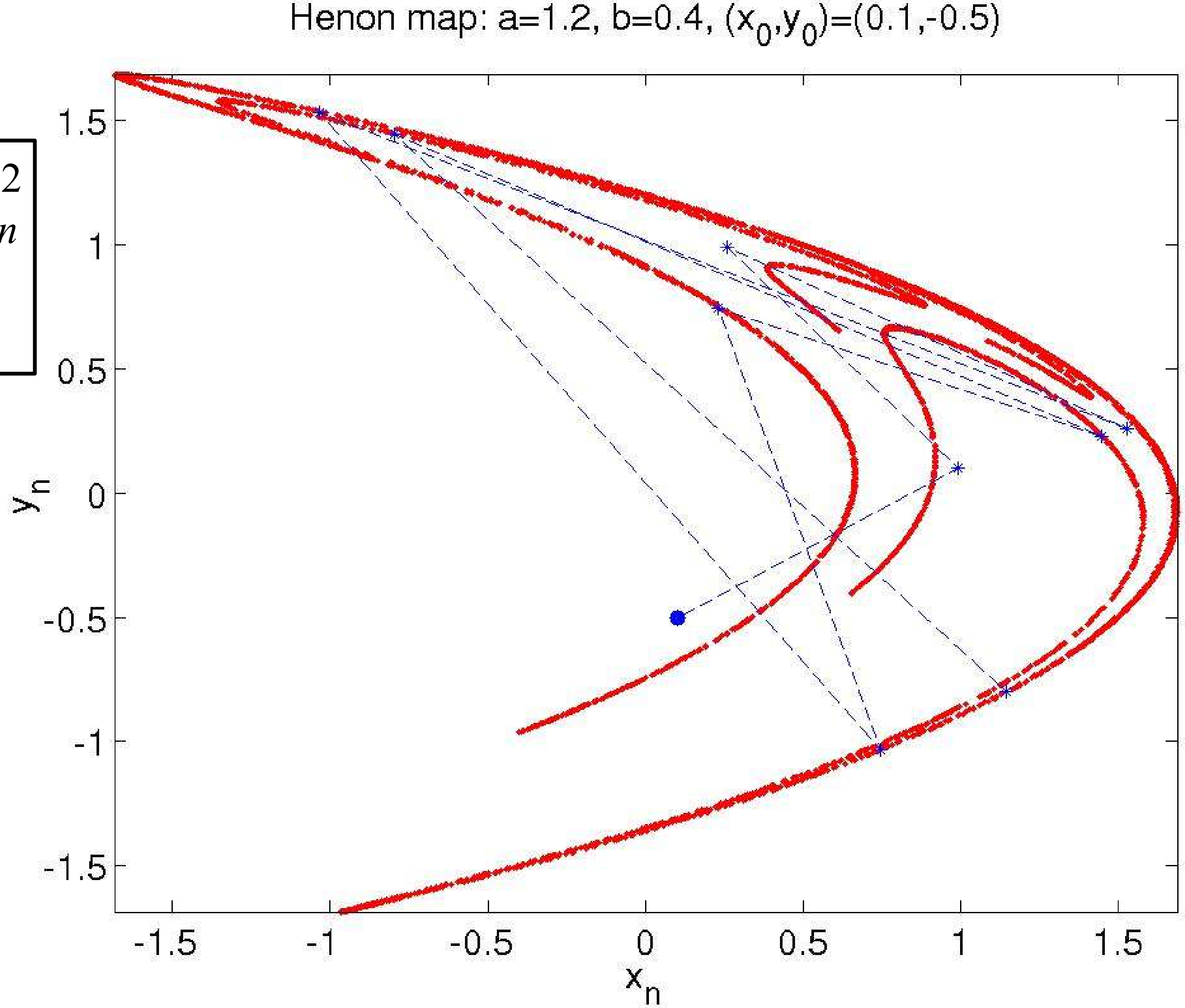


Strange attractor

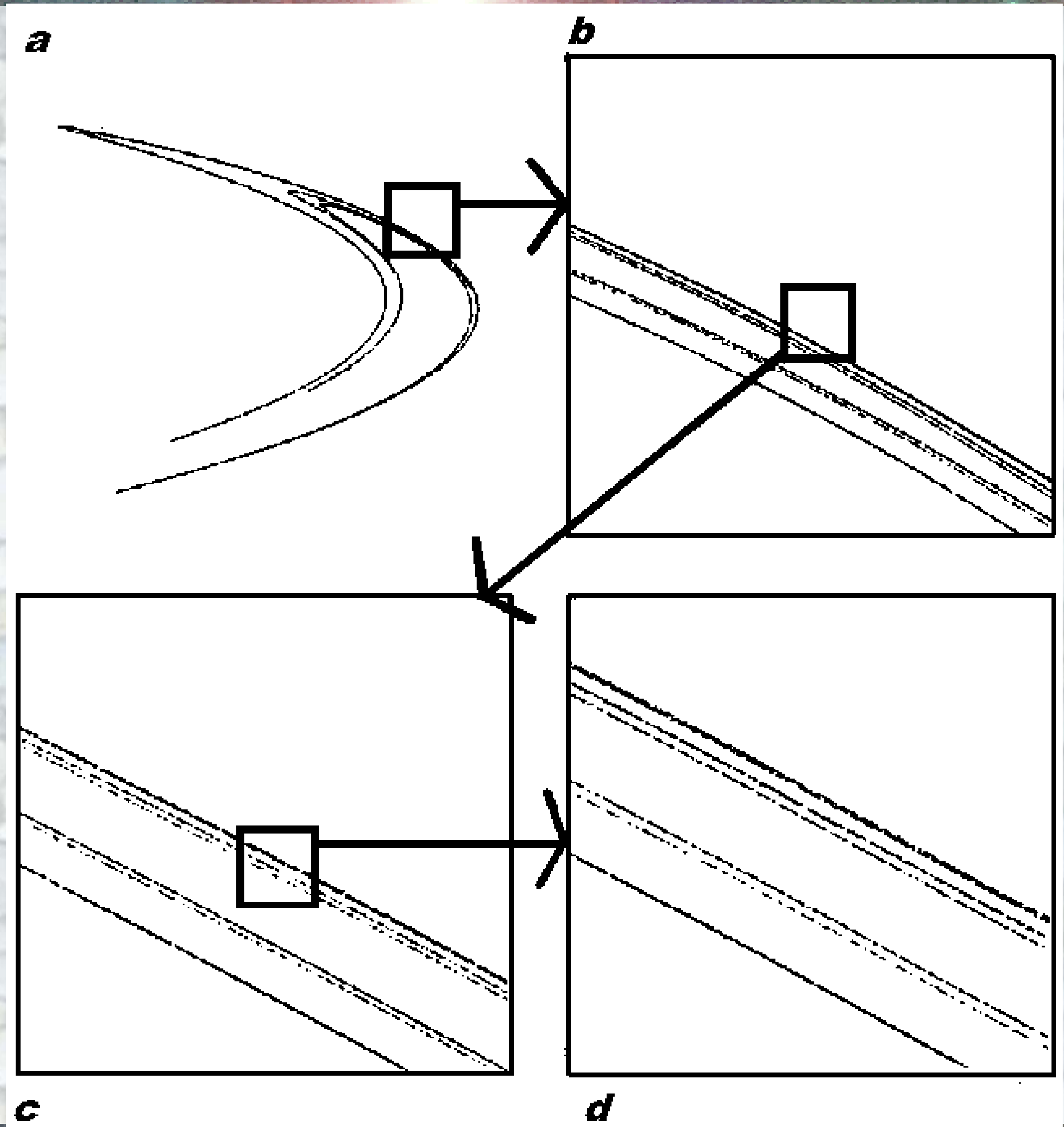
Hénon map

$$x_{n+1} = (y_n + 1) - \boldsymbol{a} x_n^2$$
$$y_{n+1} = \boldsymbol{b} x_n$$

Michel Hénon

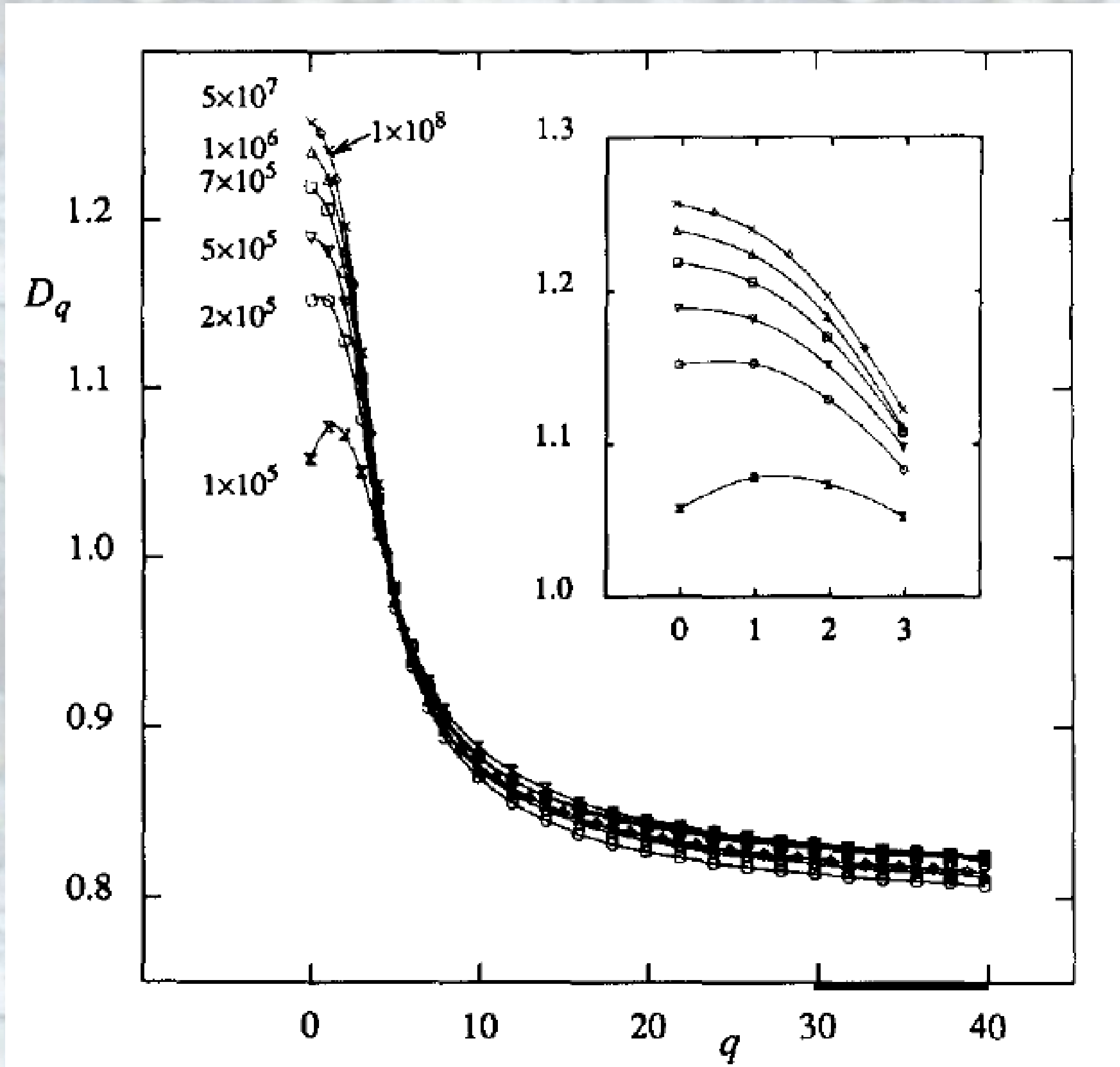


Strange attractor



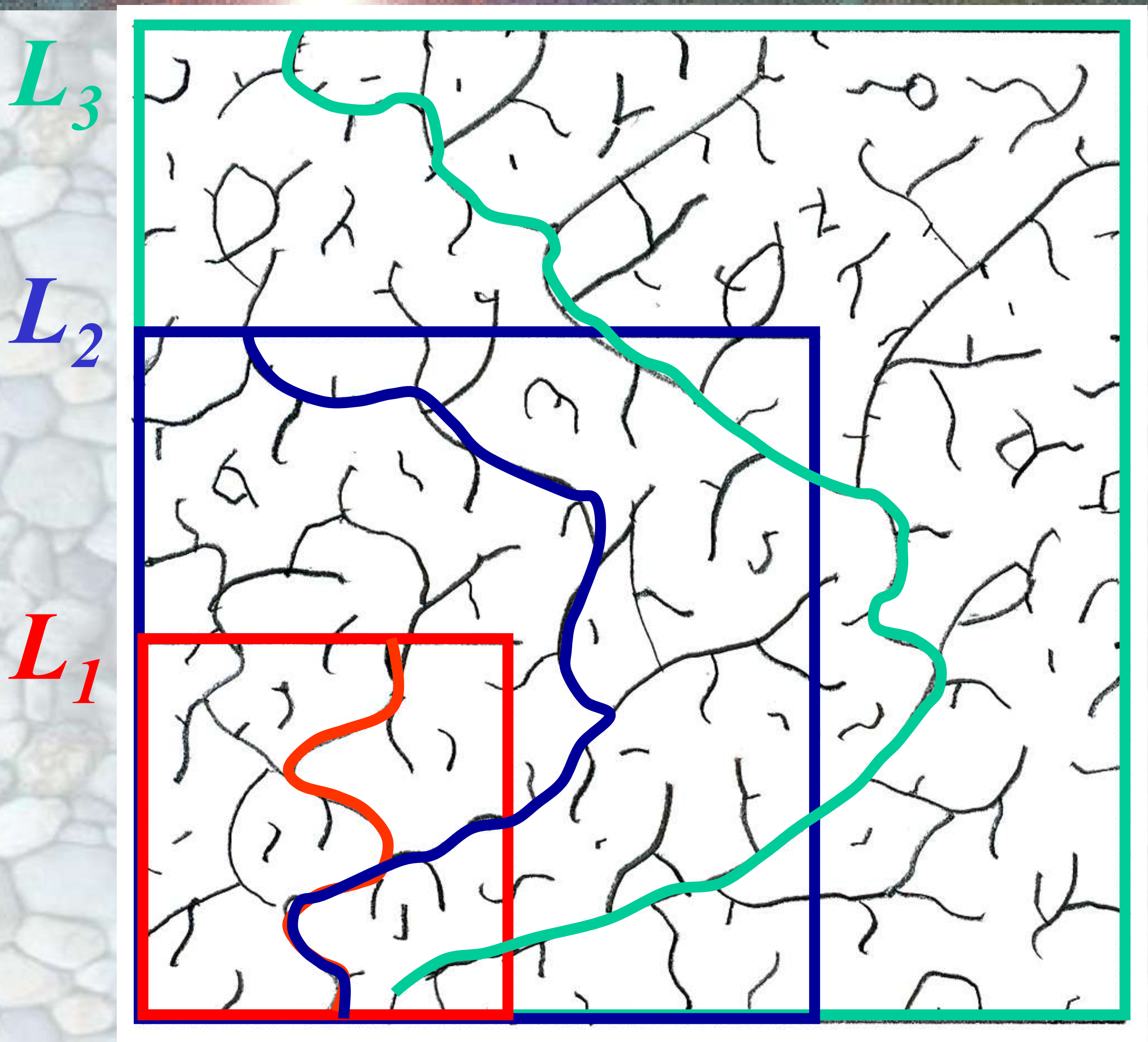
self-similar

Hénon Map



5×10^7
1×10^8
1×10^6
7×10^5
5×10^5
2×10^5
1×10^5
1.2
1.1
1.0
0.9
0.8
D_q
0
10
20
30
40
q
1.3
1.2
1.1
1.0
0
1
2
3

Volatile fractal

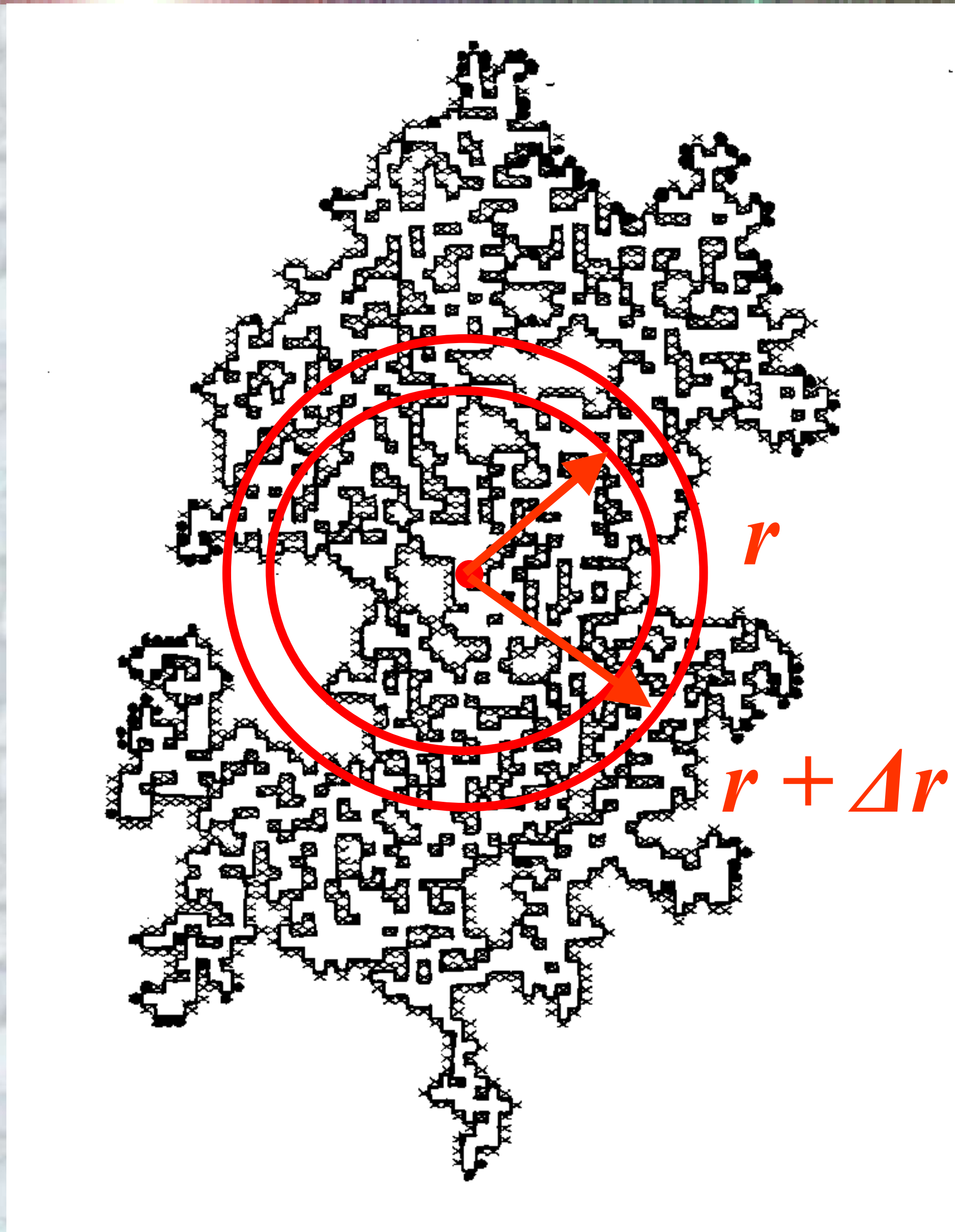


Correlation function

The correlation function $g(r)$ for percolation describes the connectivity and is defined as the probability that an occupied site is connected to a site at distance r. This is equivalent to the probability that the two sites belong to the same cluster.

The correlation length ξ is the characteristic length of the exponential decay of the correlation function.

Calculate $g(r)$



Correlation length ξ

If one just analyses one cluster

connectivity correlation function $g(r) = c(r)$

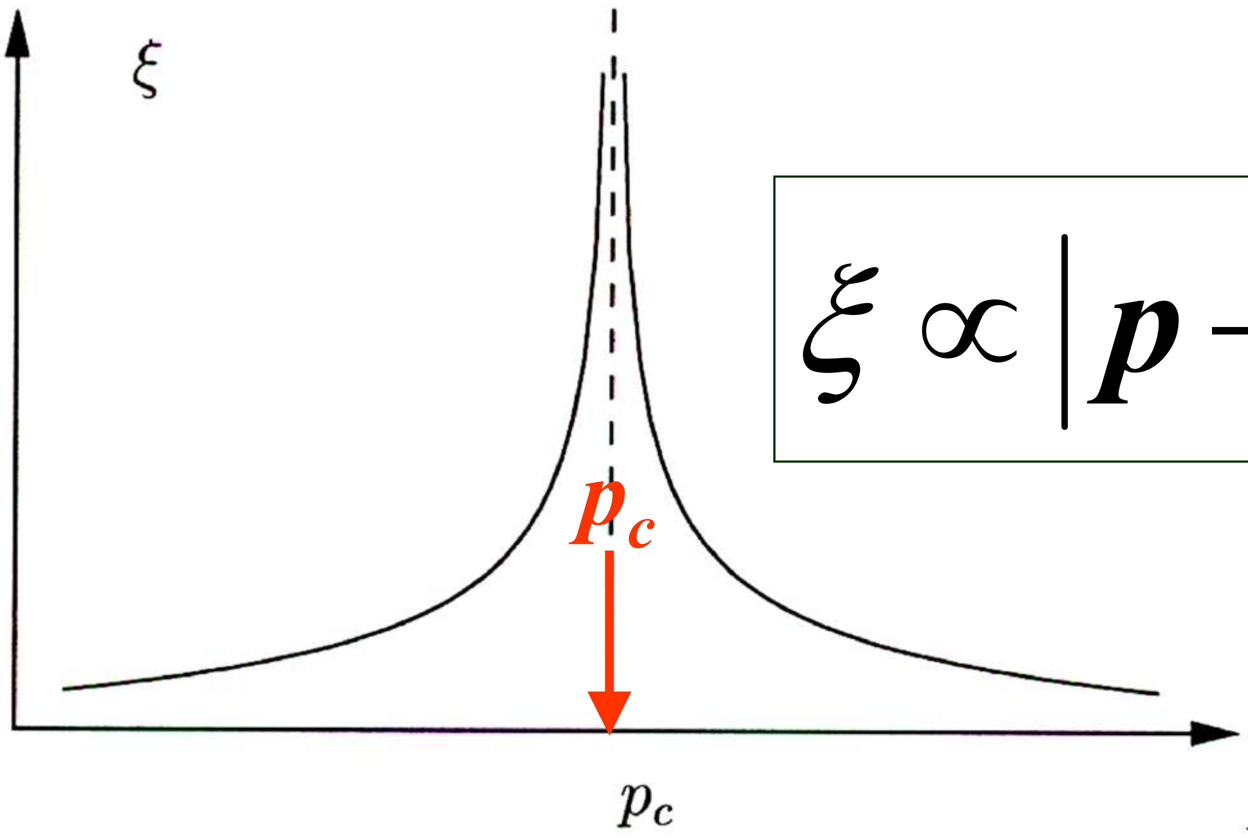
$$g(r) = \frac{\Gamma(d/2)}{2\pi^{d/2} r^{d-1} \Delta r} \left[M(r+\Delta r) - M(r) \right]$$

$$g(r) \propto C + e^{-\frac{r}{\xi}} \quad \text{with} \quad C = 0 \quad \text{for} \quad p < p_c$$

For $p < p_c$ the correlation length ξ is proportional to the radius of a typical cluster.

Correlation length ξ

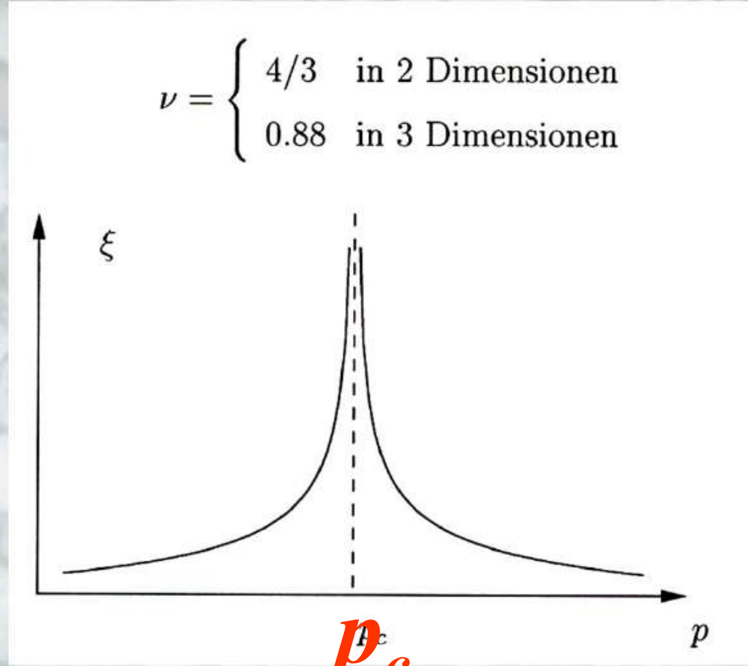
$$\nu = \begin{cases} 4/3 & \text{in 2 Dimensionen} \\ 0.88 & \text{in 3 Dimensionen} \end{cases}$$



$$\xi \propto |p - p_c|^{-\nu}$$

Correlation length ξ

$$\xi \propto |p - p_c|^{-\nu}$$



p_c

at p_c:

$$g(r) \propto r^{-(d-2+\eta)}$$

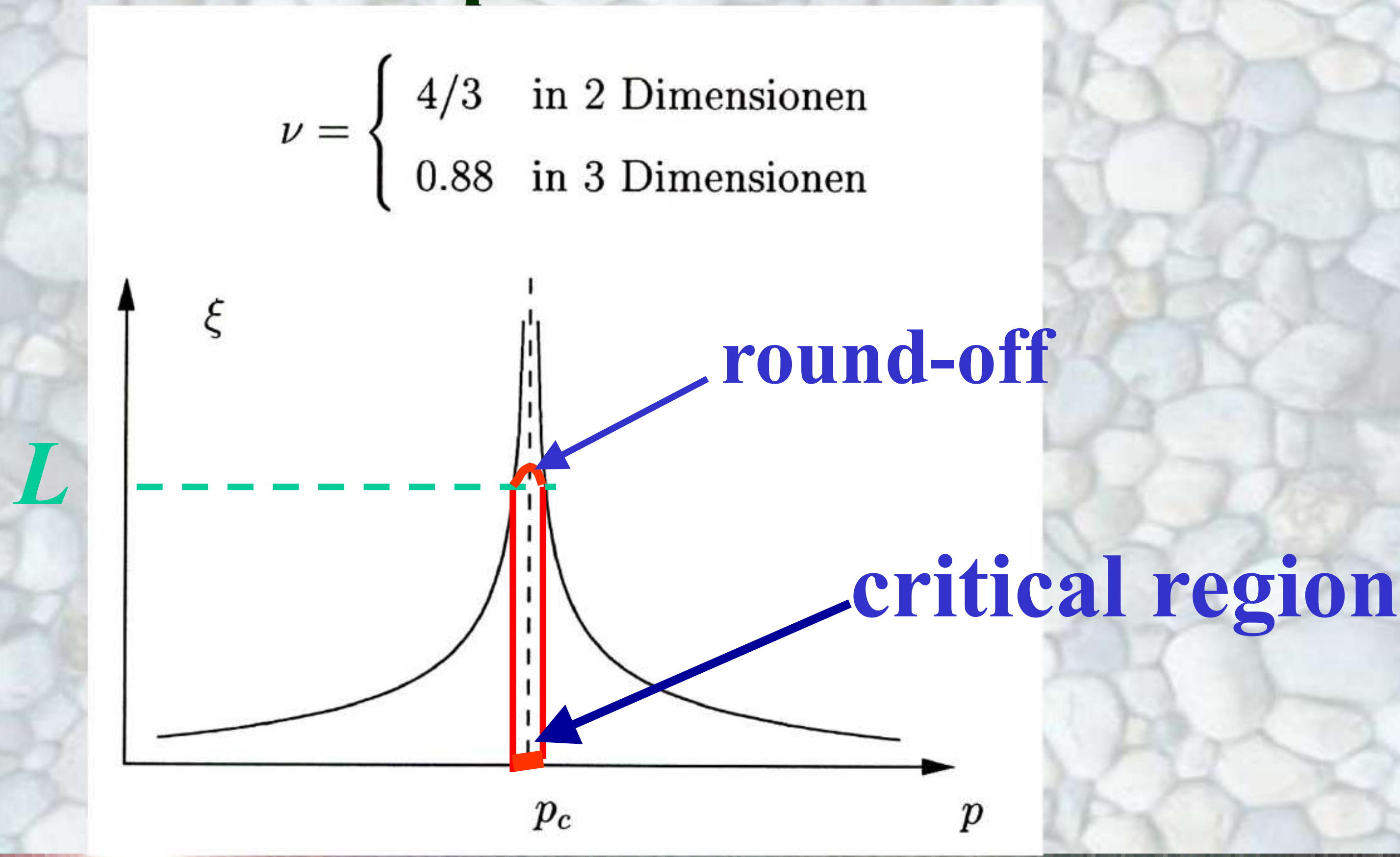
$\eta =$	5/24	$2d$
$\eta \approx$	-0.05	$3d$

Finite size effects

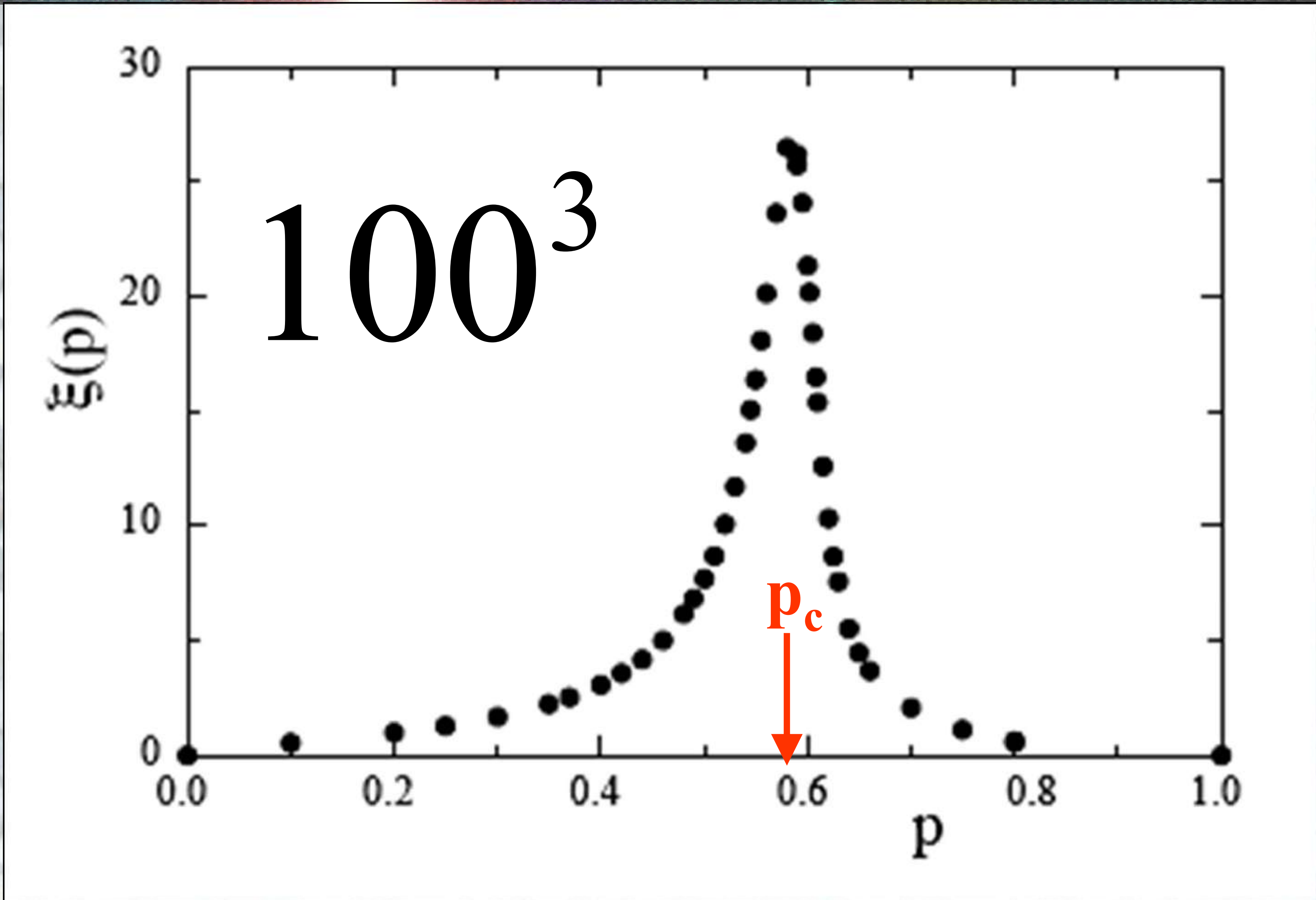
problem when:

system size L < correlation length ξ

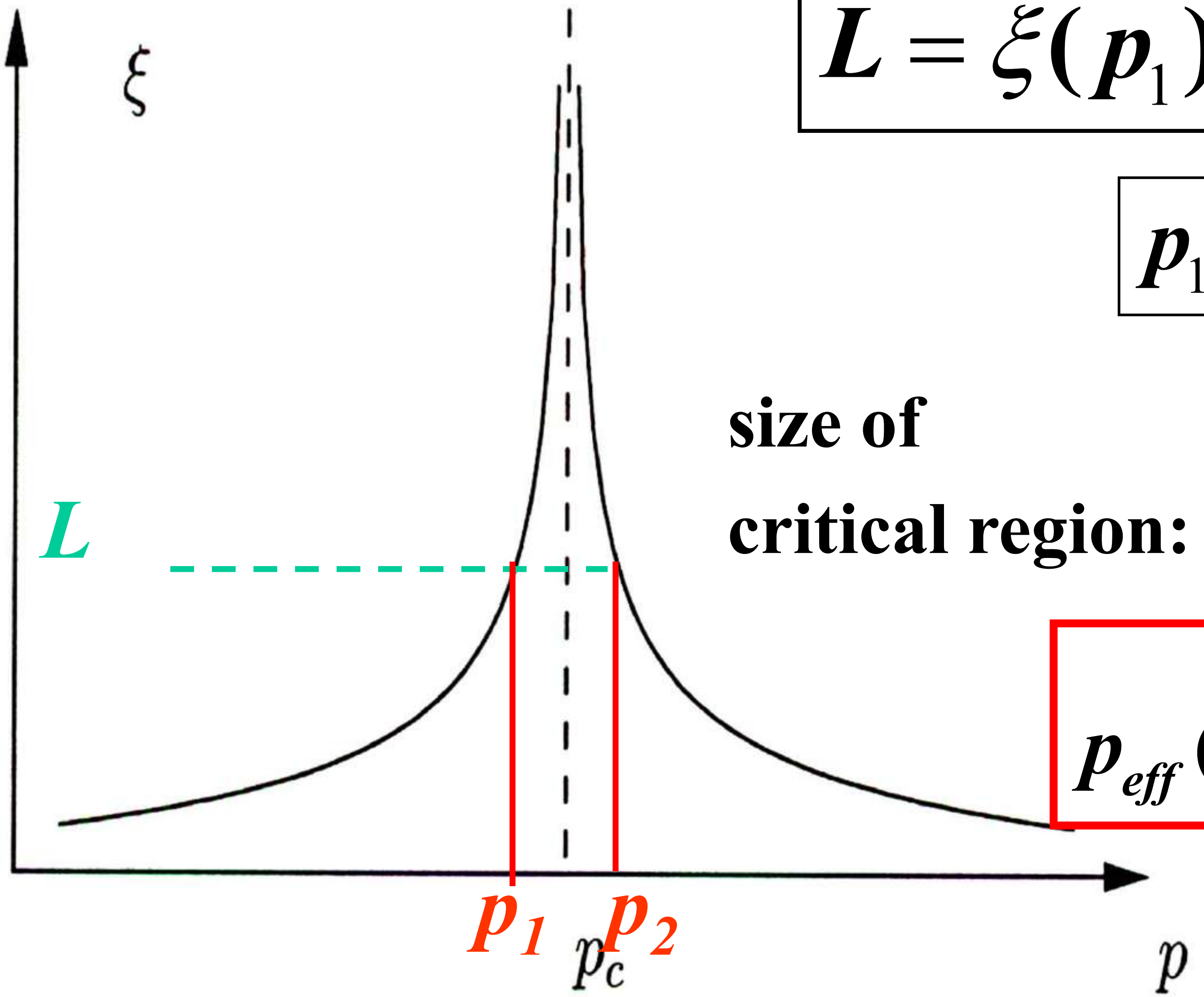
i.e. close to the critical point:



Round-off in correlation length ξ



Finite size effects



$$L = \xi(p_1) \propto (p_1 - p_c)^{-\nu}$$

$$p_1 - p_2 \propto p_1 - p_c$$

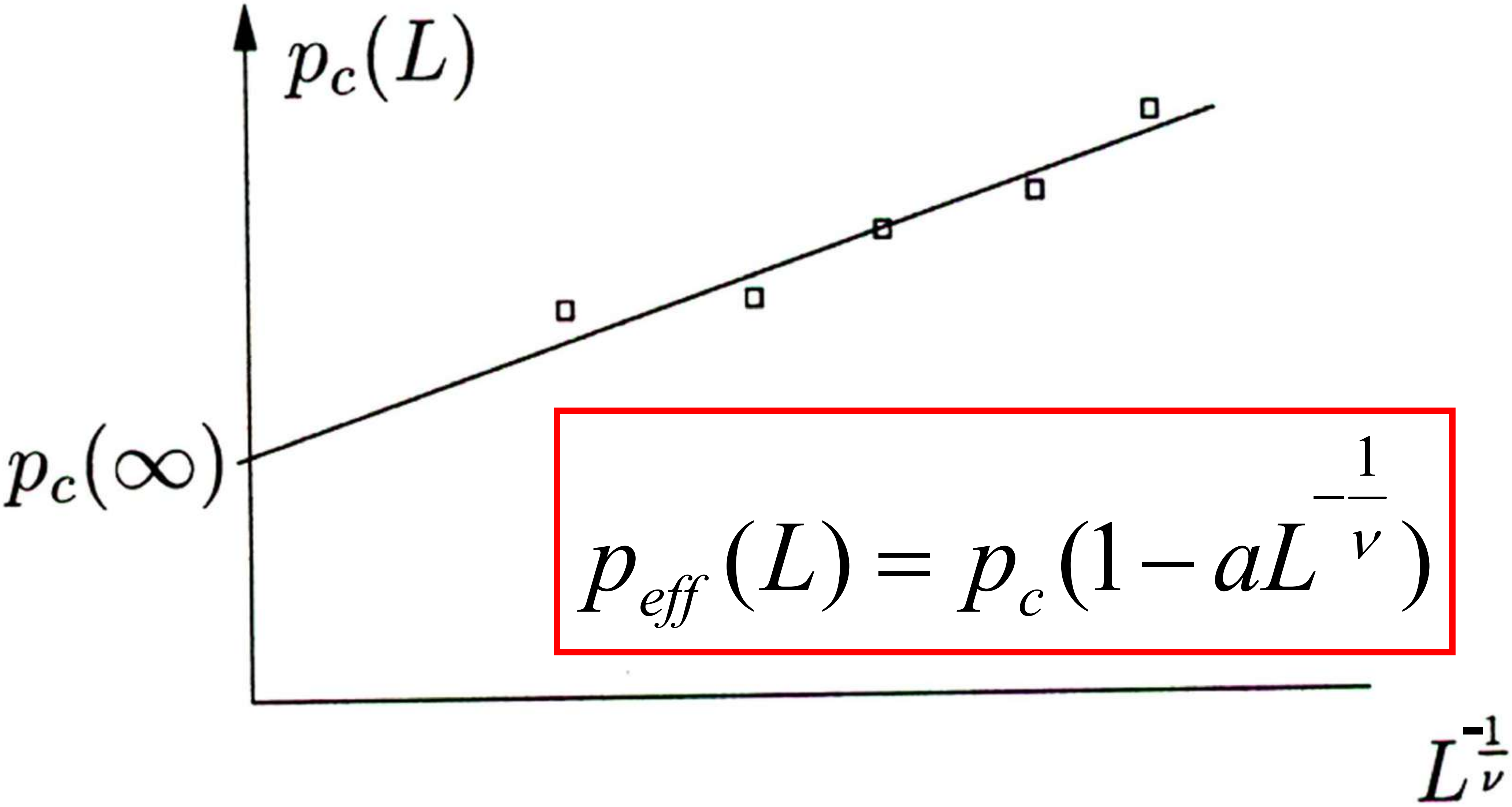
size of critical region:

$$p_1 - p_2 \propto L^{-\frac{1}{\nu}}$$

$$p_{eff}(L) = p_c(1 - aL^{-\frac{1}{\nu}})$$

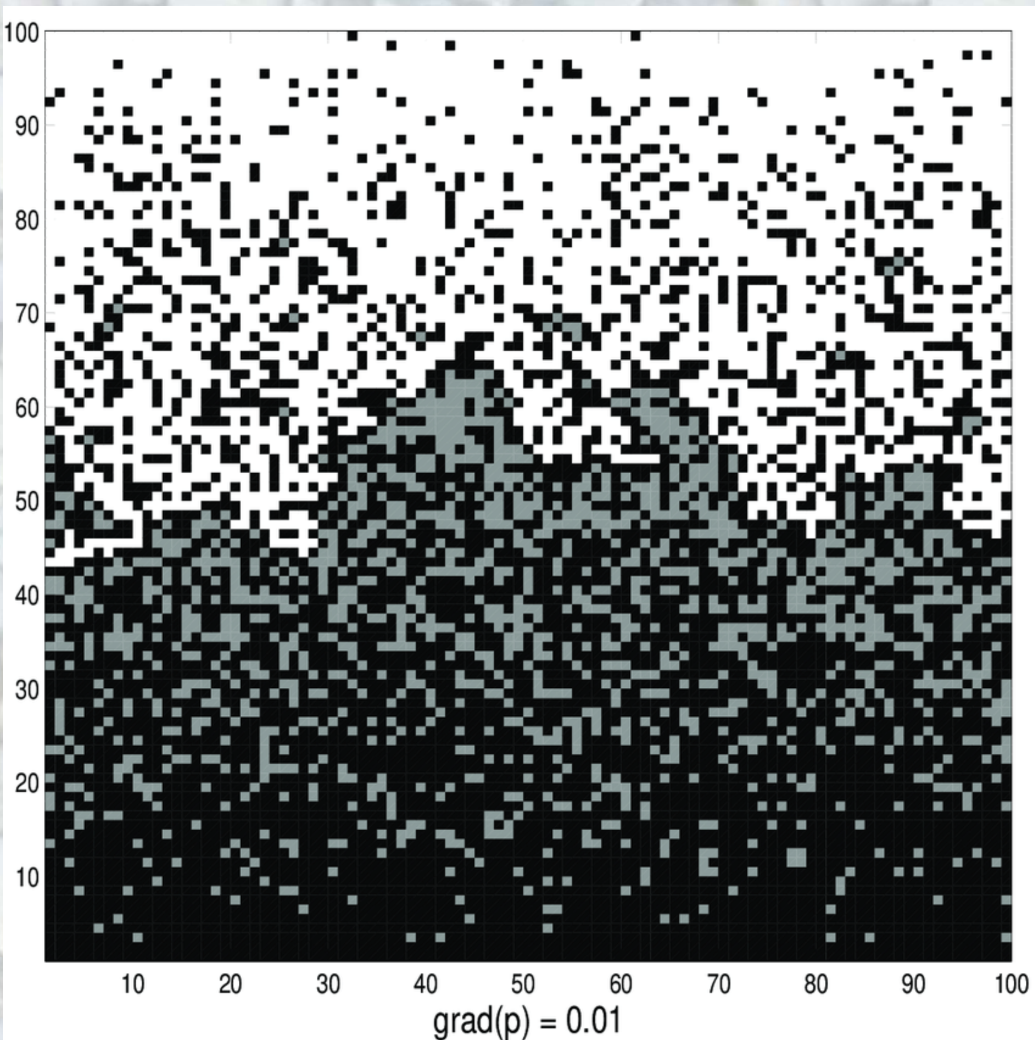
Apply finite size dependence

Extrapolation to infinite size



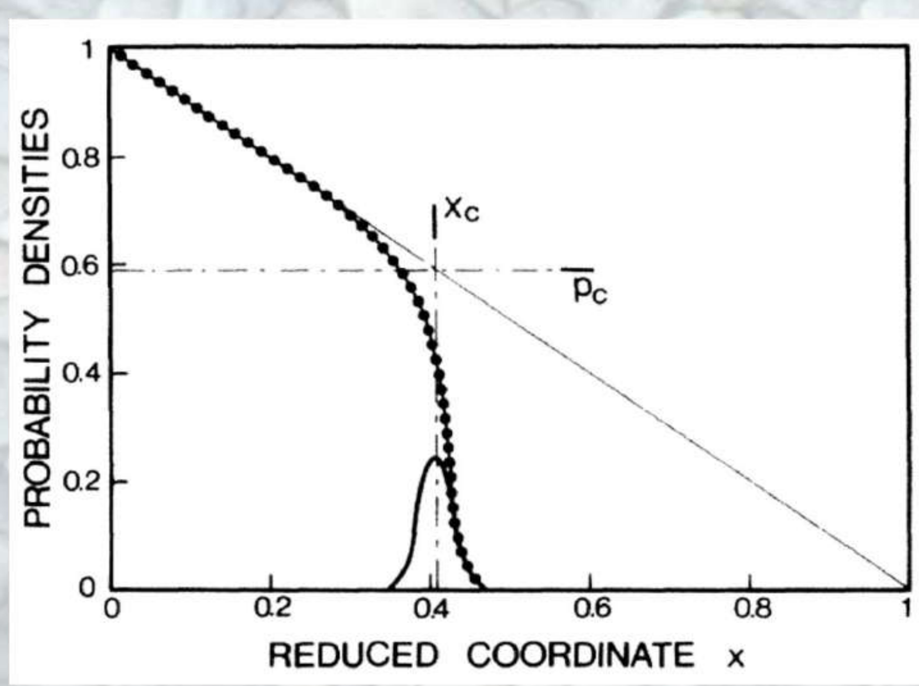
$$p_{eff}(L) = p_c(1 - aL^{-\frac{1}{\nu}})$$

Gradient percolation



1 x

$$p(x) = 1 - x|\nabla p|$$

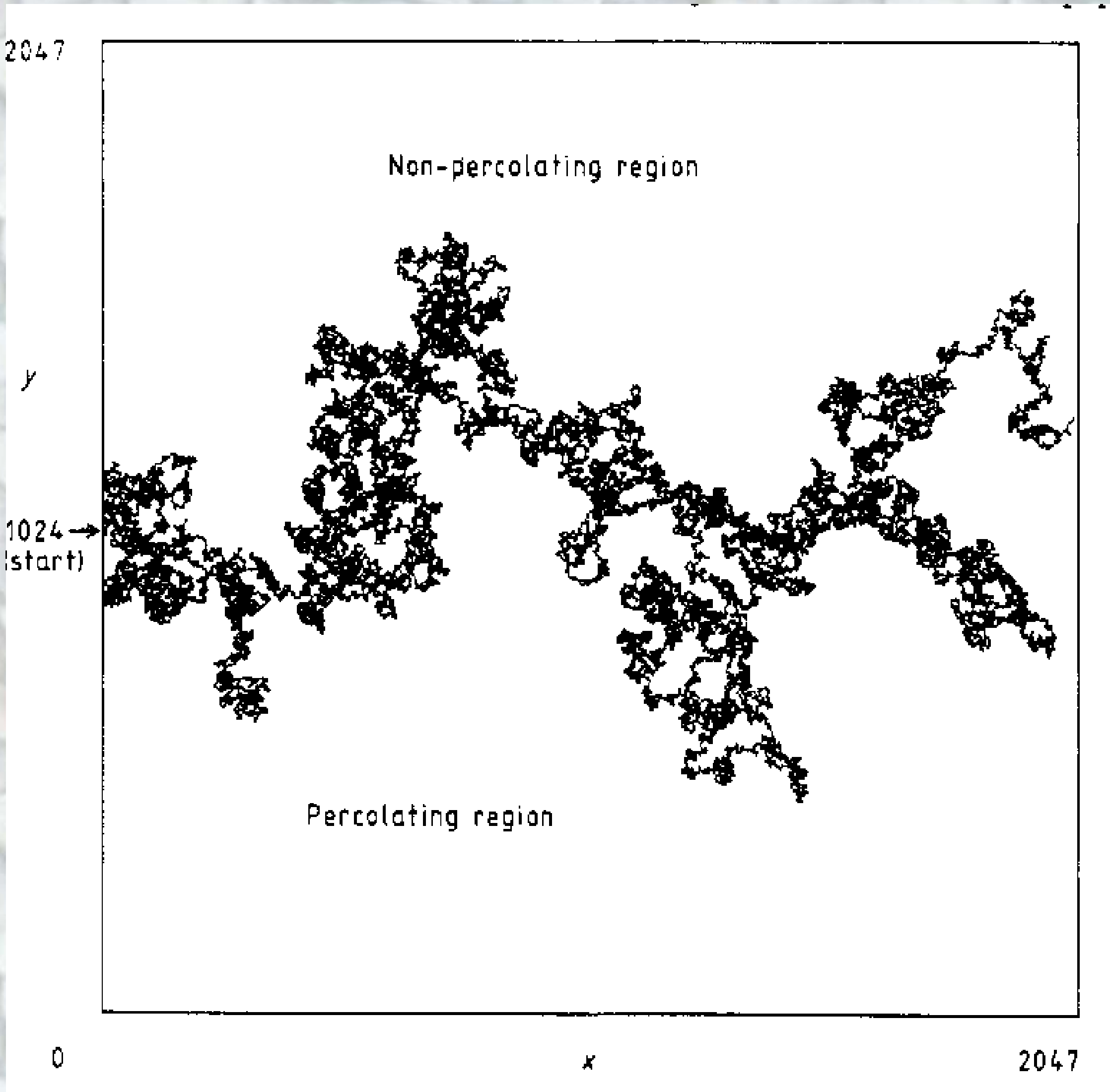


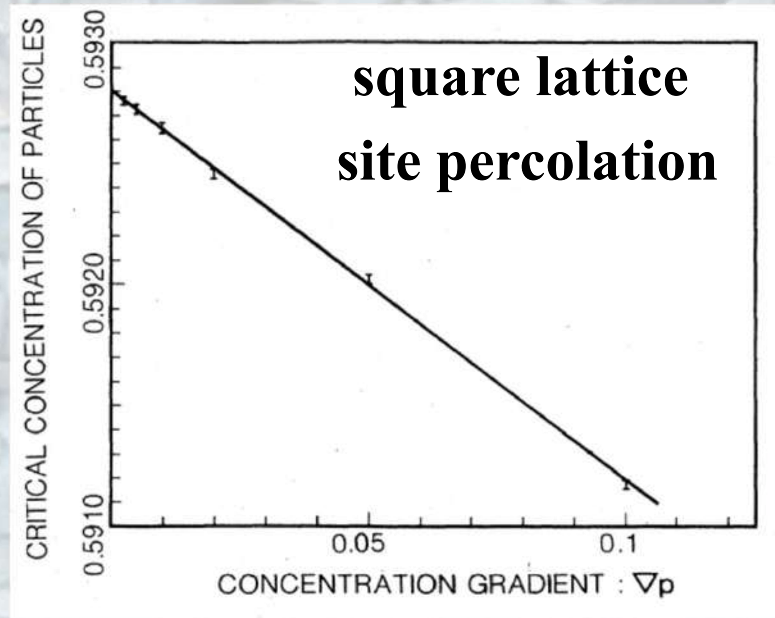
0

$$\sigma_{fB} \propto |\nabla p|^{-0.57} = \frac{\nu}{1+\nu}$$

M Rosso, JF Gouyet, B Sapoval, J. Phys. Lett. 46, L149 (1985)

Gradient percolation



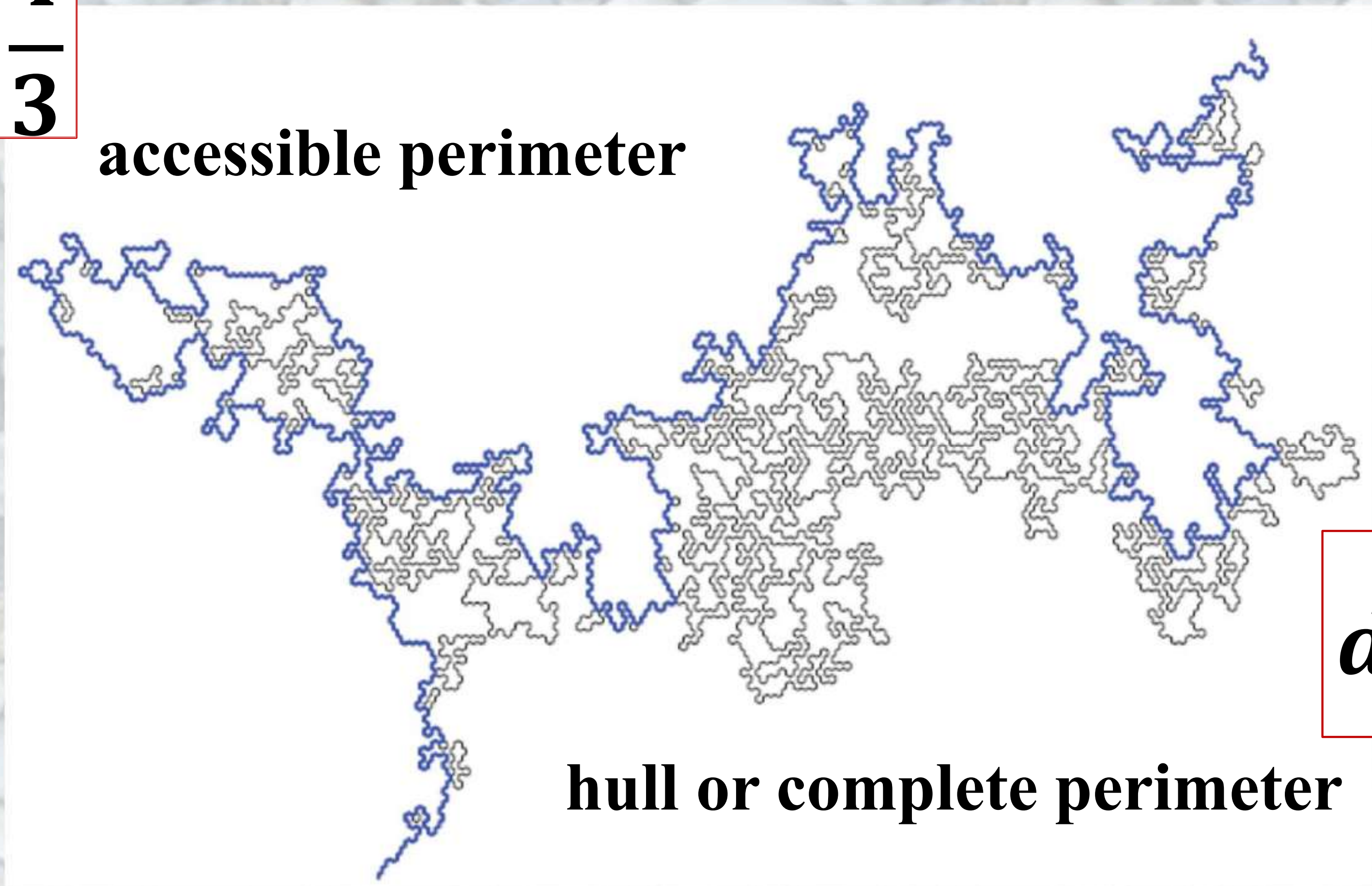


Perimeters

in two dimensions

Duplantier 1999

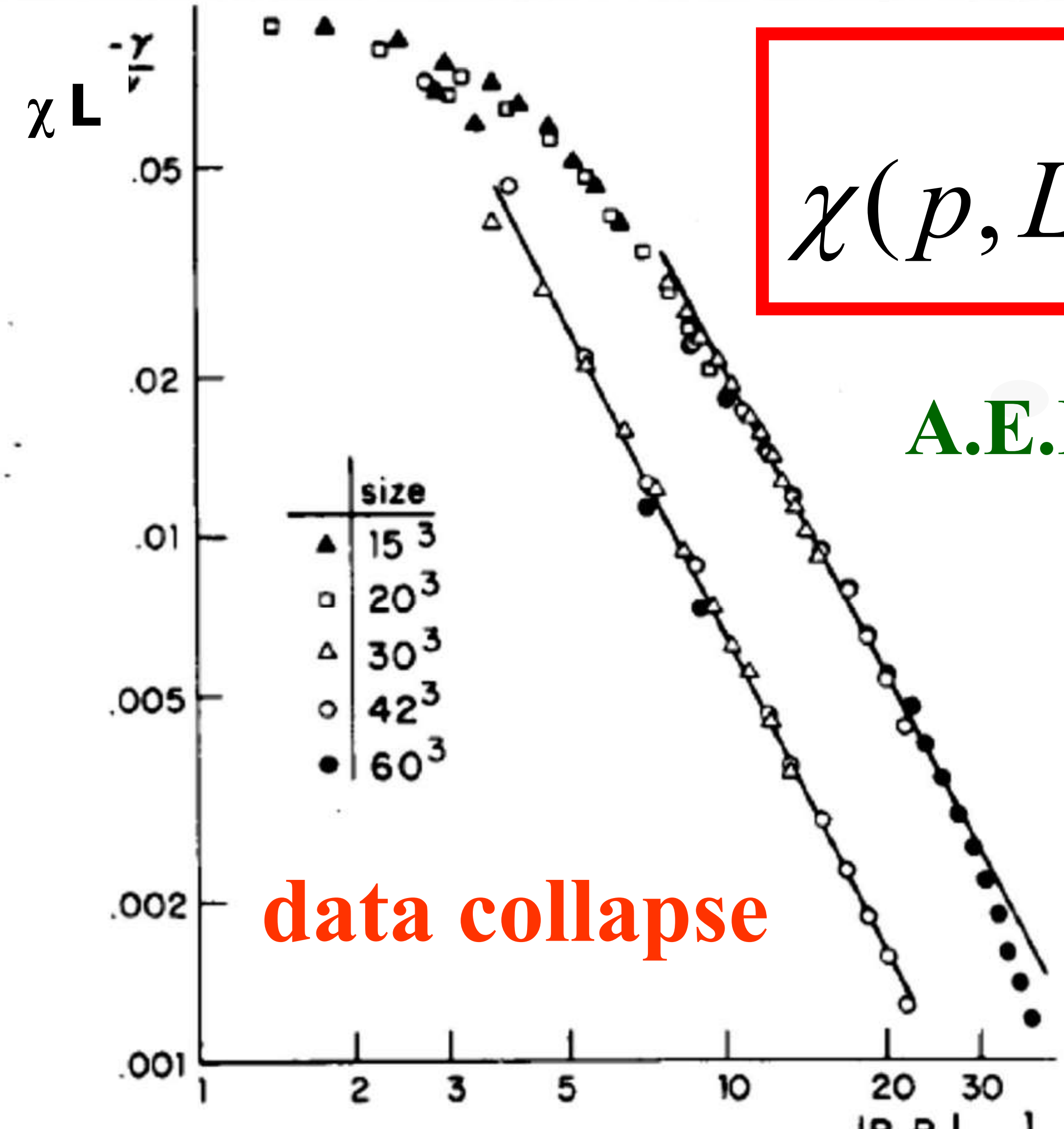
$$d_f = \frac{4}{3}$$



Saleur, 1987

$$d_f = \frac{7}{4}$$

Finite size scaling for χ



HH et al. (1982)

$$\chi(p,L) = L^{\frac{\gamma}{\nu}} \aleph_\chi[(p-p_c)L^{\frac{1}{\nu}}]$$

A.E.Ferdinand and M.E Fisher (1967)

at p_c:

$$\chi_{\max}(L) \propto L^{\frac{\gamma}{\nu}}$$

Finite size scaling of OP

fraction of sites in spanning cluster (OP): $P \propto (p - p_c)^{\beta}$

finite size scaling: $P(p,L) = L^{-\frac{\beta}{\nu}} \aleph_P[(p - p_c)L^{\frac{1}{\nu}}]$

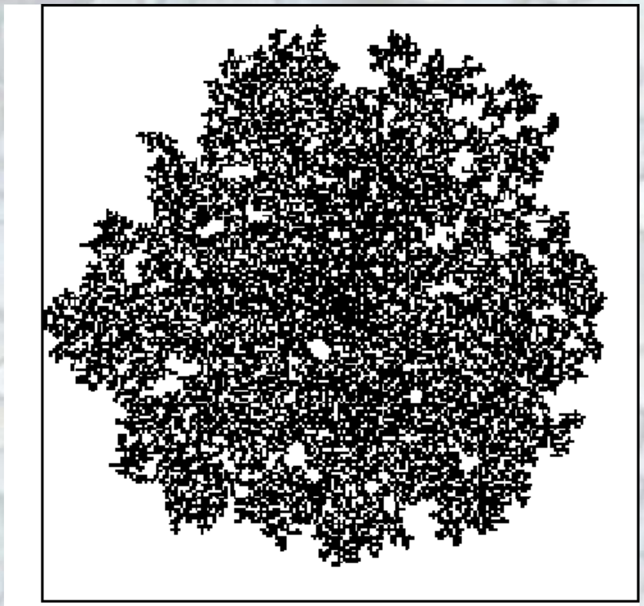
at p_c: $P \propto L^{-\frac{\beta}{\nu}}$

$M_\infty \propto L^{d_f}$

$M_\infty \propto PL^d \propto L^{-\frac{\beta}{\nu}+d} \propto L^{d_f}$

fractal dimension: $d_f = d - \frac{\beta}{\nu}$

Making individual clusters



1. **Leath algorithm**

   for any value of *p*

2. **Invasion percolation**

   only at p_c

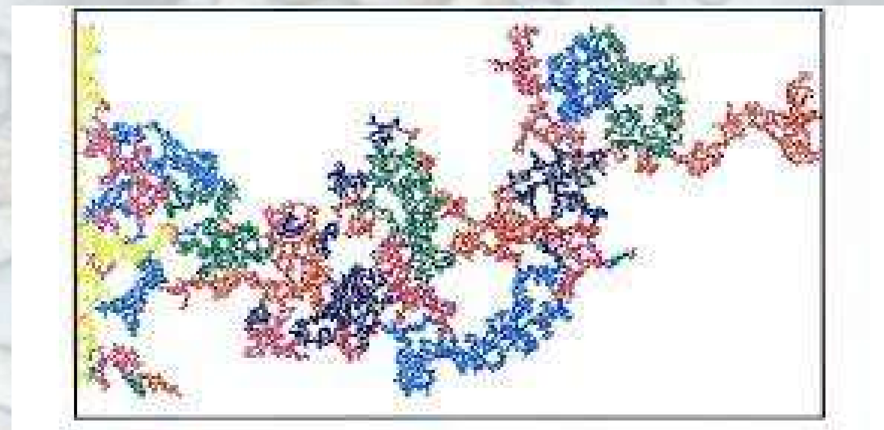
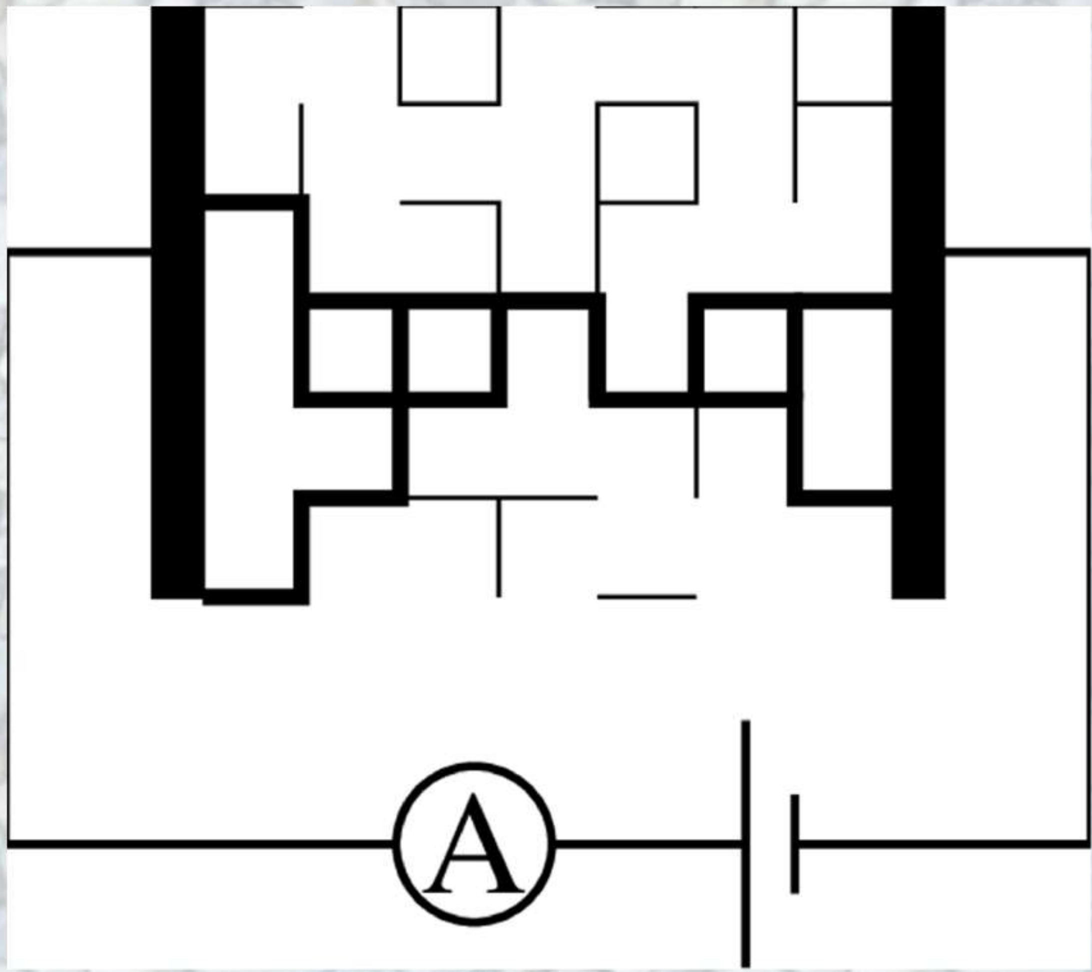


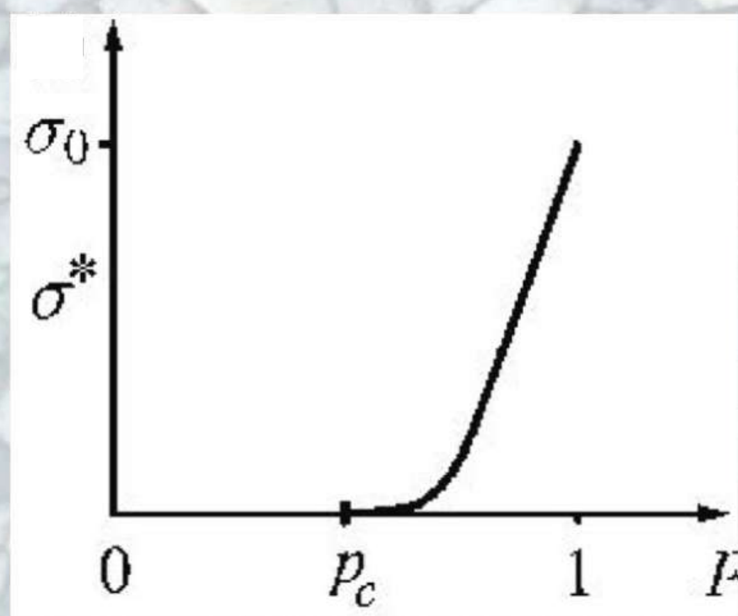
FIG. 1: Invasion percolation with trapping on a 300×600 lattice. The invader (colored) enters from sites on the left-hand edge and the defender (white) escapes through the right-hand edge. At breakthrough the invader first reaches the right-hand edge and has invaded 31482 sites. Different colors (left to right on color scale) indicate sites added within successive time intervals = 2131.

Dynamics on percolation clusters

Occupied bonds are conductors of conductivity = 1 and empty bonds are insulators.



electrical conductivity



$$\sigma \sim (p - p_c)^t \sim L^{t/\nu}$$

$t \approx 1.2993 \pm 0.0015$ **in 2d**

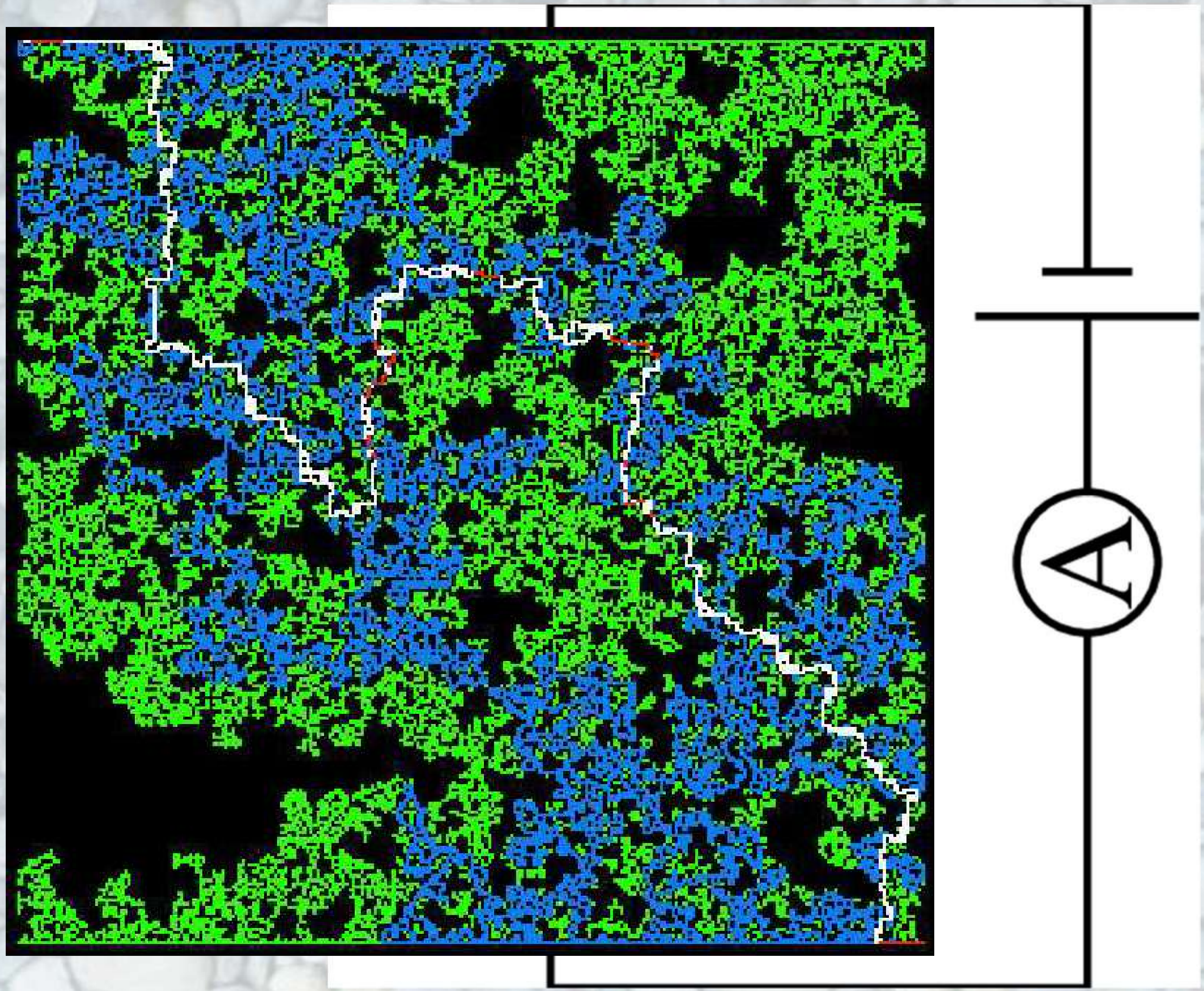
$t \approx 1.81 \pm 0.04$ **in 3d**

One can also consider mixtures of conductor – superconductor.

obtained with PERCOLA

Dynamics on percolation clusters

at p_c



Dynamics on percolation clusters

Remove the dangling ends from the IIC and you get the backbone = current carrying subset.

$$d_{BB} = 1.64333 \pm 0.0001 \text{ in 2d}$$

$$d_{BB} = 1.875 \pm 0.003 \text{ in 3d}$$

Red bonds or cutting bonds disrupt the current if removed.

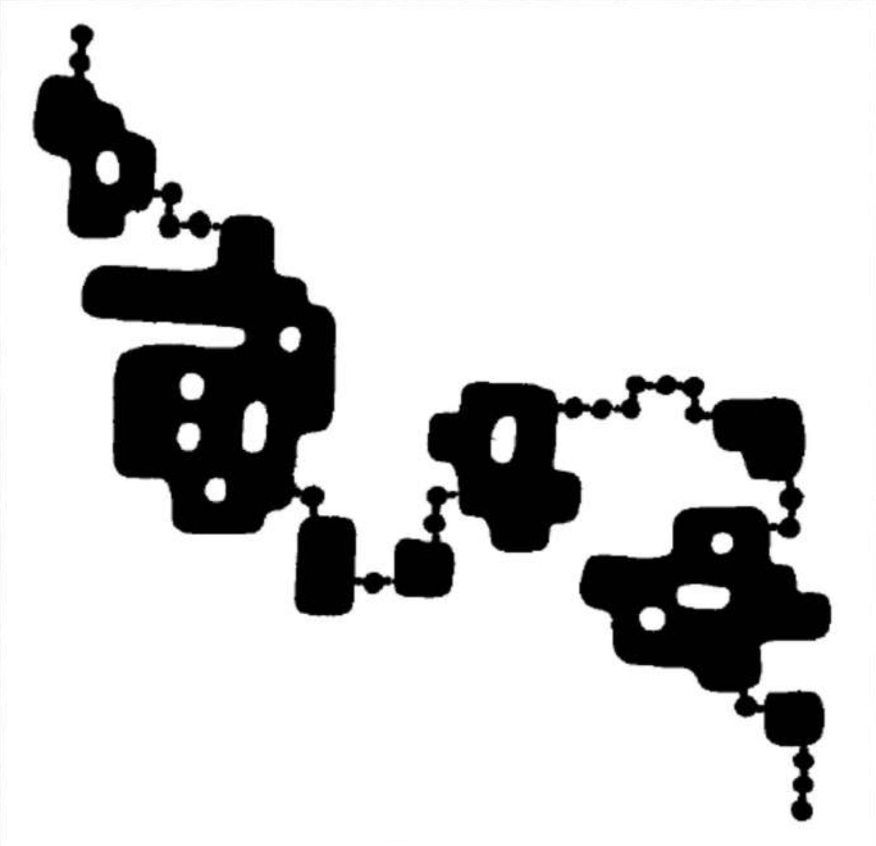
Their fractal dimension is 1/ν (A. Coniglio, 1981).

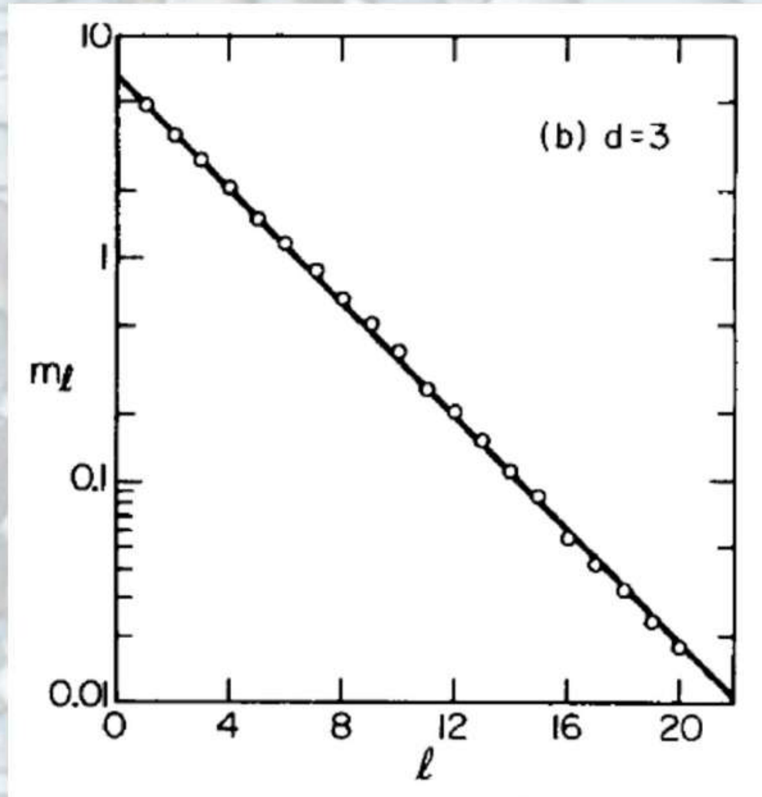
In two dimensions 1/ν = ¾ < 1

Structure of the backbone

Backbone is a necklace of randomly chosen blobs.

probability to find a string of ℓ red bonds





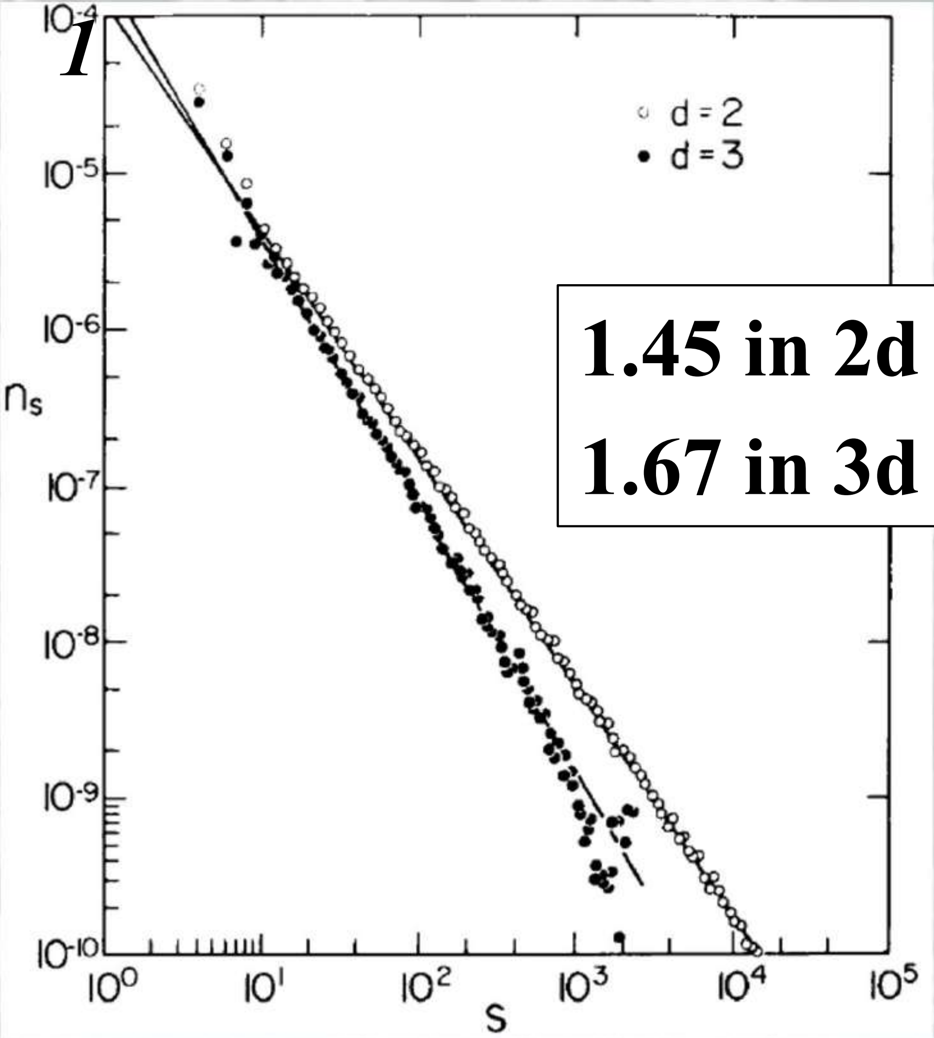
HJH and HE Stanley, Phys. Rev. Lett. 53, 1121 (1984)

Structure of the backbone

blob size distribution

$$n_s(L) \sim s^{-\tau} f(s/L^{d_f^{BB}})$$

$$\tau - 1 = d/d_f^{BB}$$



HJH and HE Stanley, Phys. Rev. Lett. 53, 1121 (1984)

Hierarchy of critical exponents

moments of the distribution of currents *n*(*i*)

$$M_0 = \sum_{all\ bonds} n(i) \sim L^{d_{BB}}$$ **backbone**

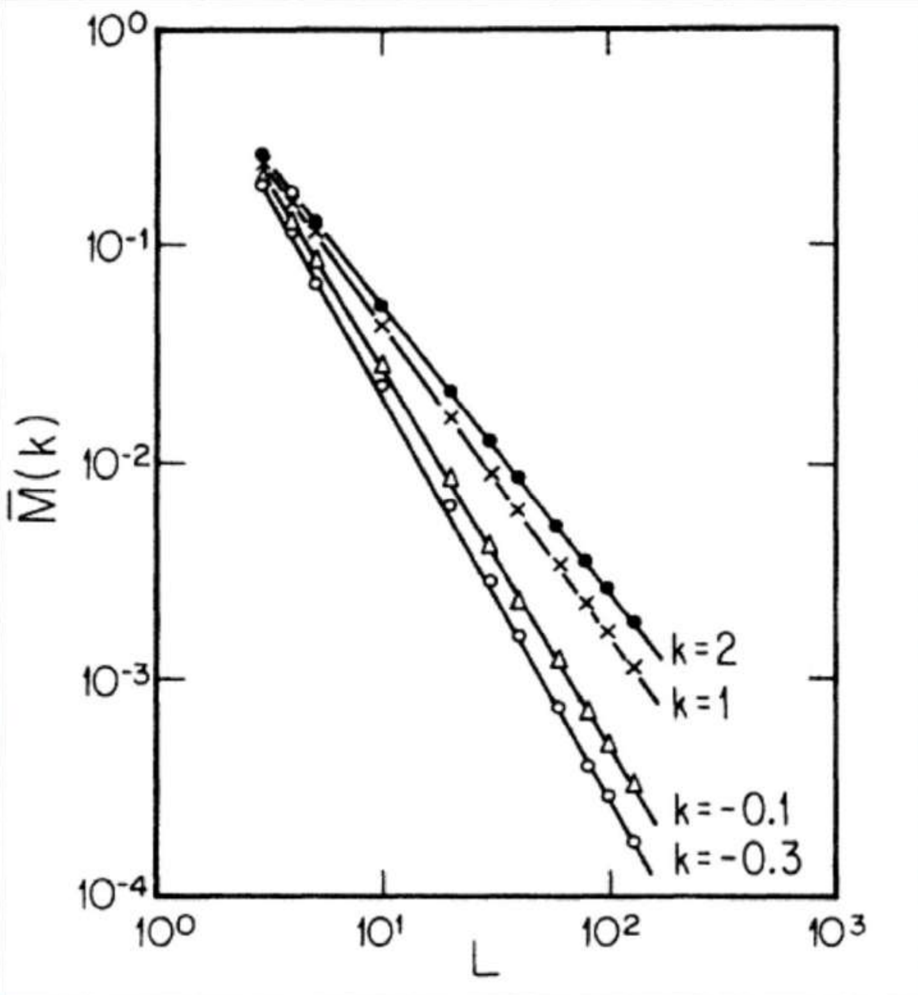
$$M_1 = \sum_{all\ bonds} i\, n(i) \sim L^{t/\nu}$$ **conductivity**

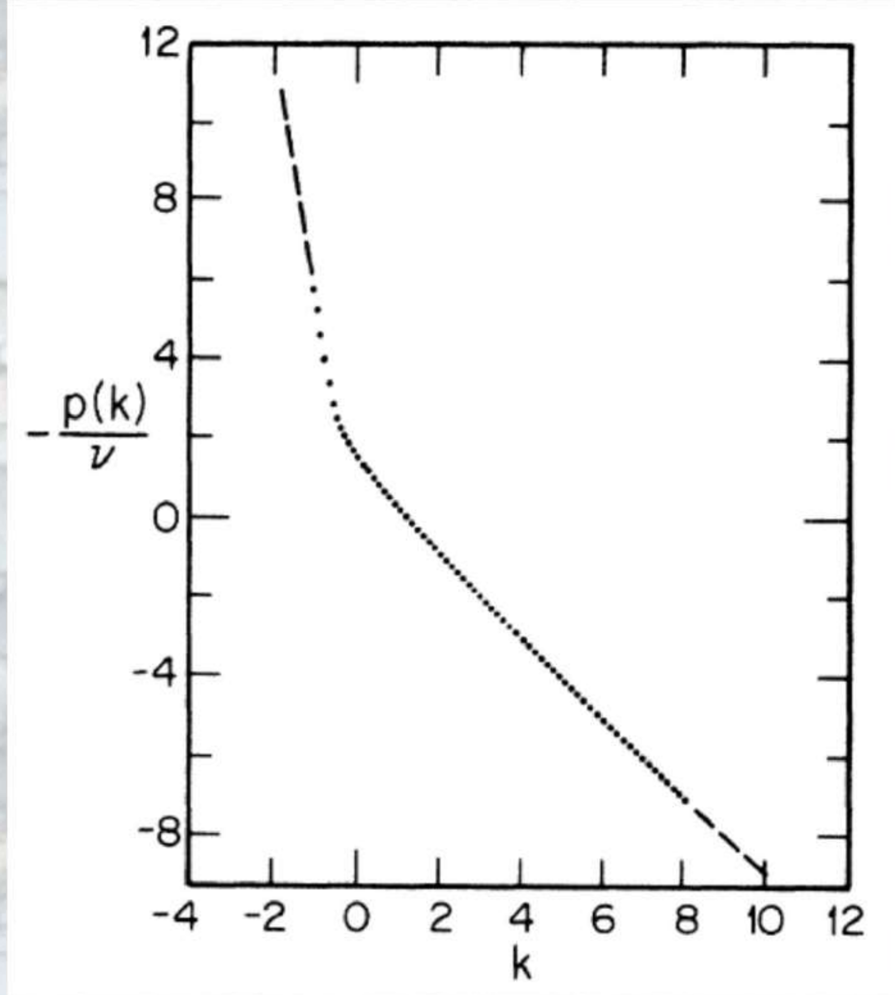
$$M_2 = \sum_{all\ bonds} i^2\, n(i) \sim L^{\zeta}$$ **flicker noise**

$$\cdots$$

$$M_\infty = \sum_{all\ bonds} i^\infty\, n(i) \sim L^{1/\nu}$$ **cutting bonds**

Multifractal current distribution





L. de Arcangelis, S. Redner, A. Coniglio, Phys. Rev. B 31, 4725 (1985)

Multifractal current distribution

V **is voltage drop at each bond**

$$M(q) = \sum_V n(V) V^q \sim L^{-\tau(q)}$$

Legendre transformation

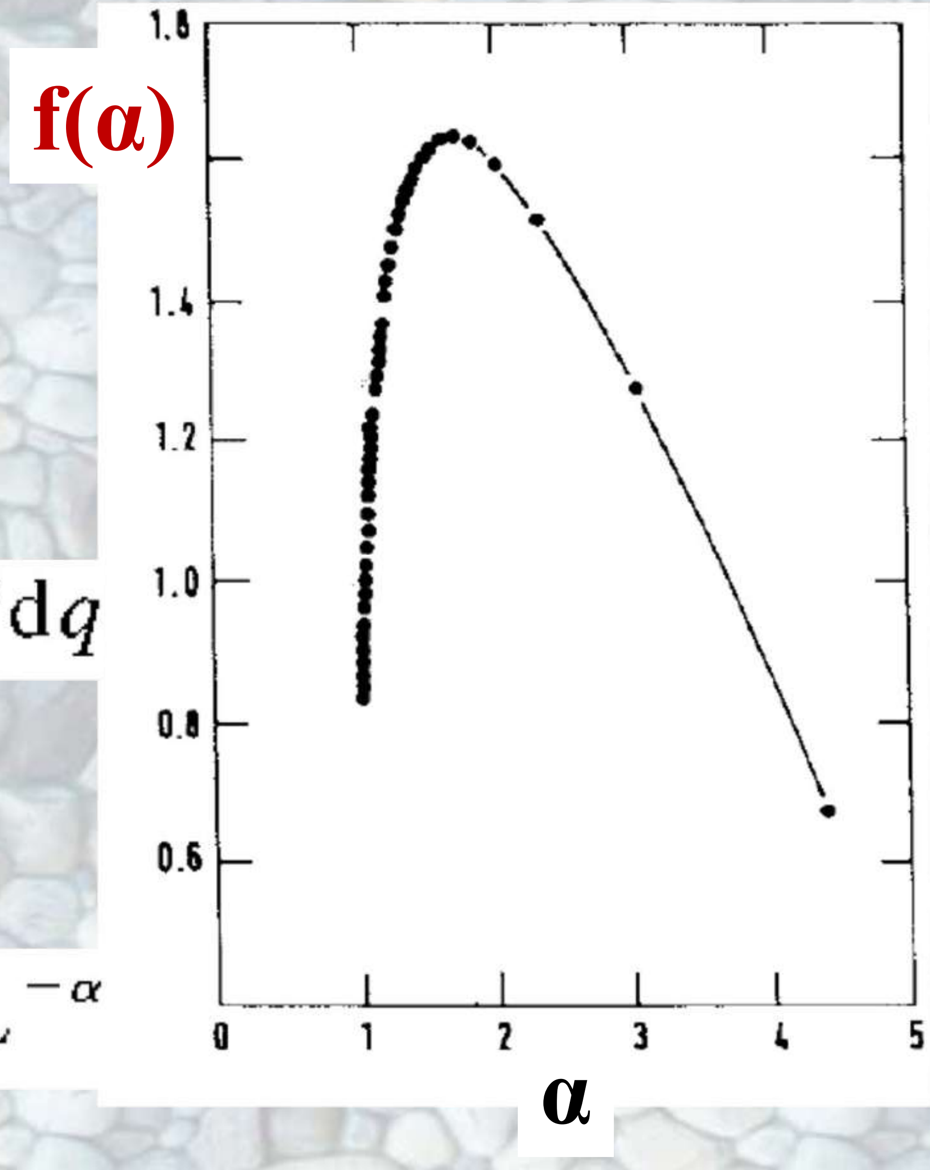
$$f(\alpha) = q\alpha - \tau(q)$$

with

$$\alpha = \mathrm{d}\tau(q)/\mathrm{d}q$$

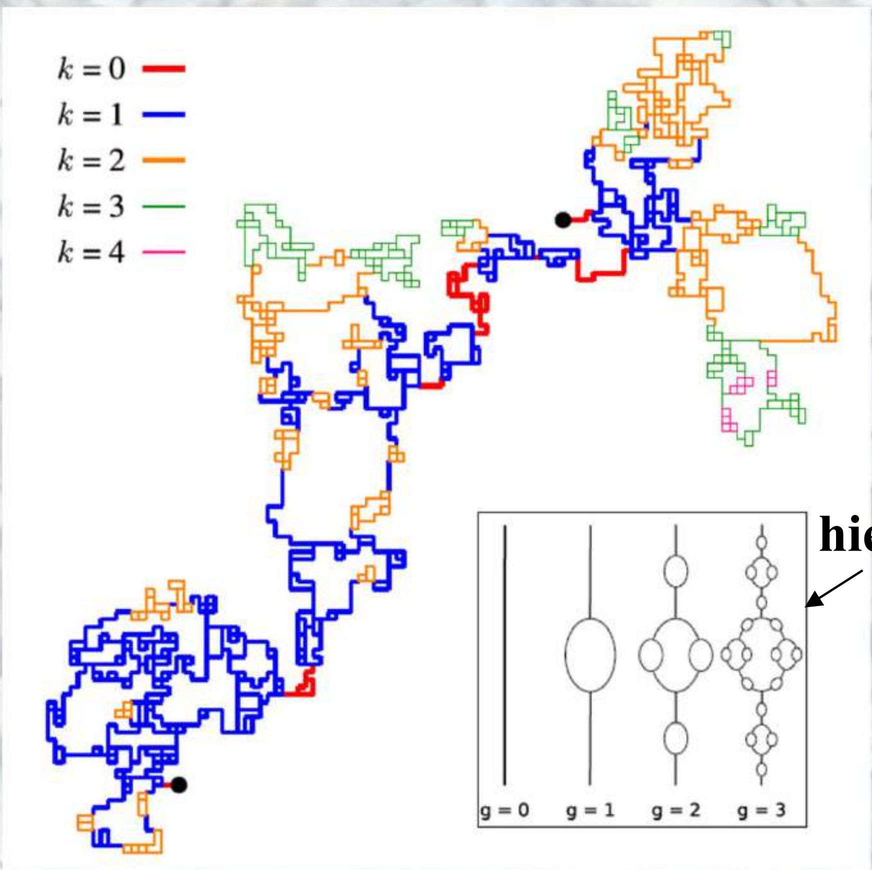
$$n(V) \sim C(\alpha) L^{f(\alpha)}$$

$$\alpha = -\ln V/\ln L, \qquad \text{i.e. } V \sim L^{-\alpha}$$



Calculating the current distribution

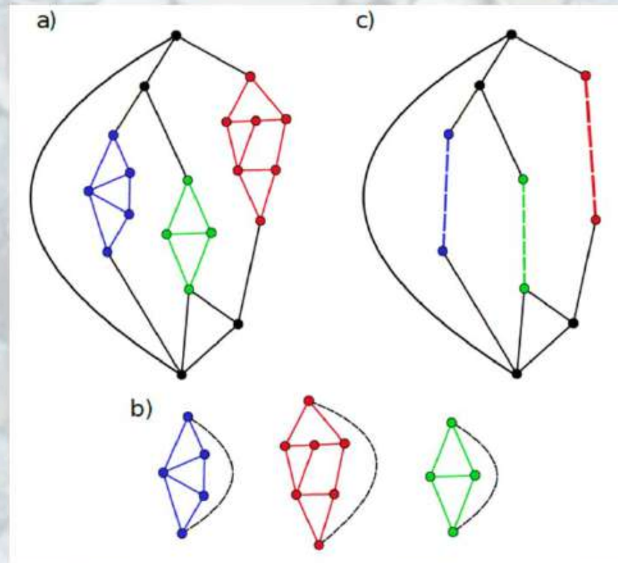
W.R. de Sena, J.S. Andrade Jr., H.J.H., A.A. Moreira



hierarchical lattice

A 3-connected component can not be decomposed in series and parallel.

decompose in 3-connected components:



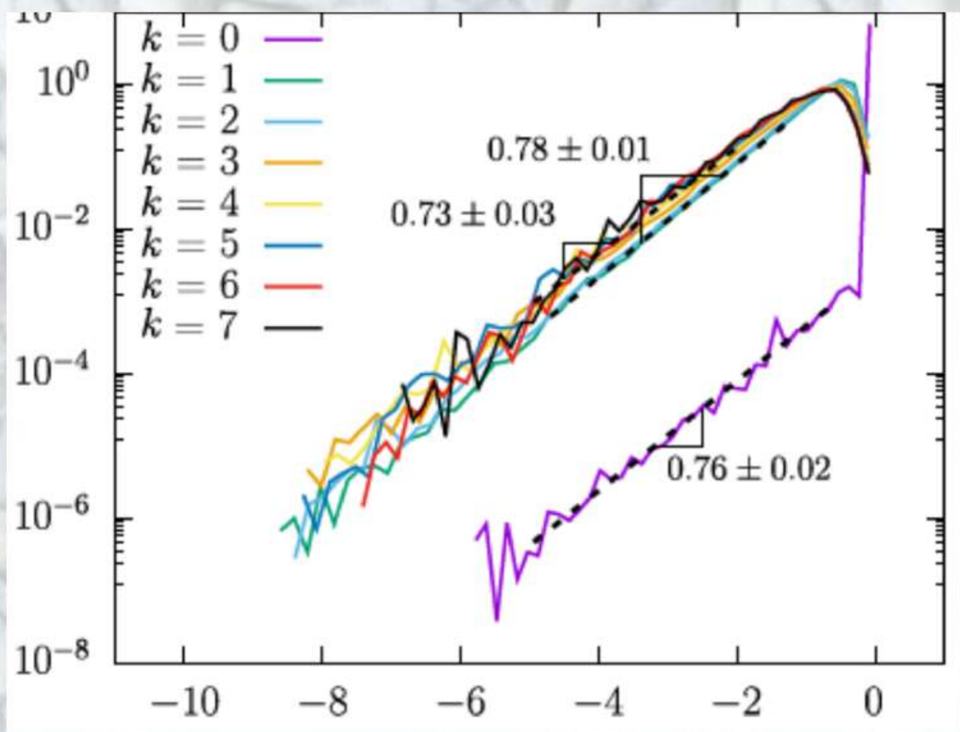
The largest 3-connected component scales like $L^{1.15}$

One can obtain local currents with precision up to 10^{-35}

Calculating the current distribution

At each level *k* each 3-connected components must be multiplied by a factor *f* to consider the corresponding reduction of the current.

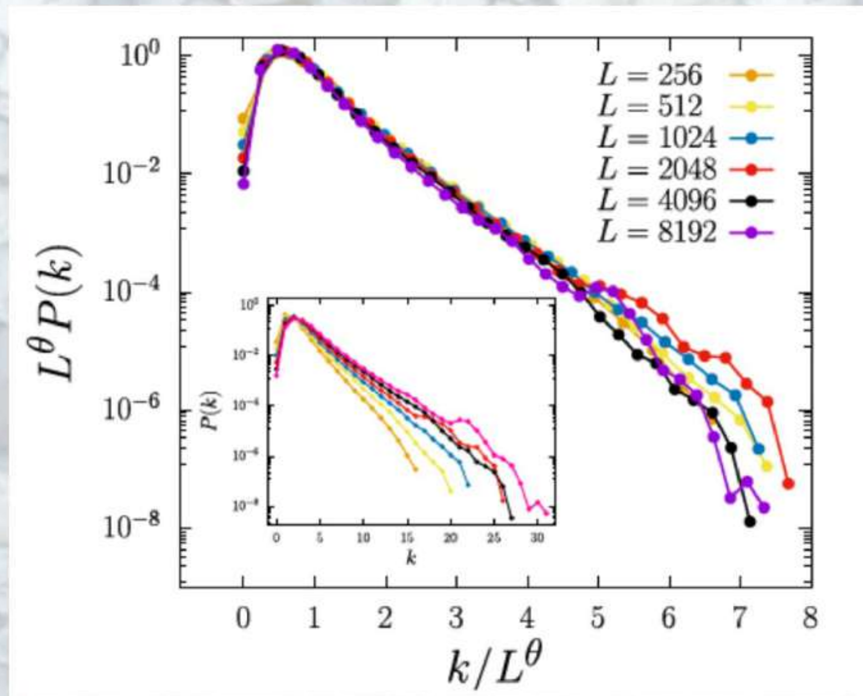
distribution of factors *f* in level *k*



power law

$$P(f) \sim f^{\alpha} \ , \quad \alpha \approx 3/4$$

distribution of levels *k*

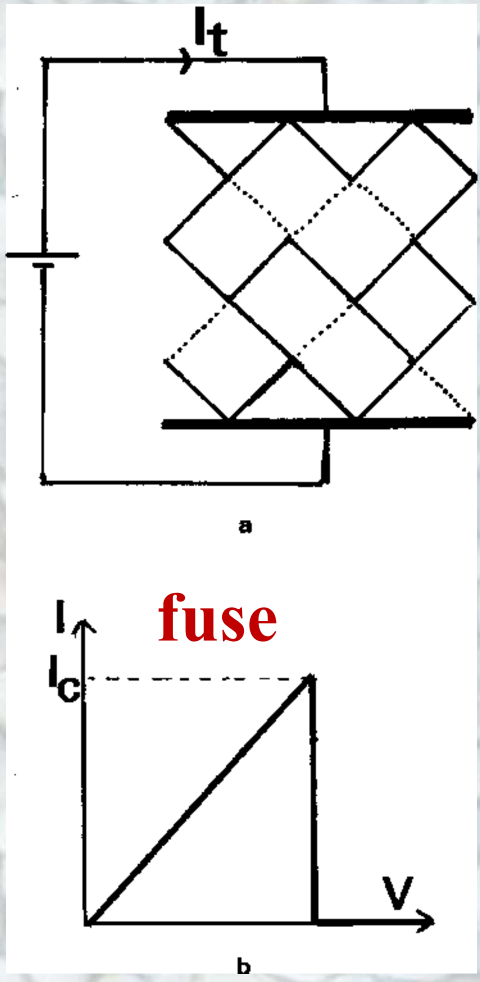


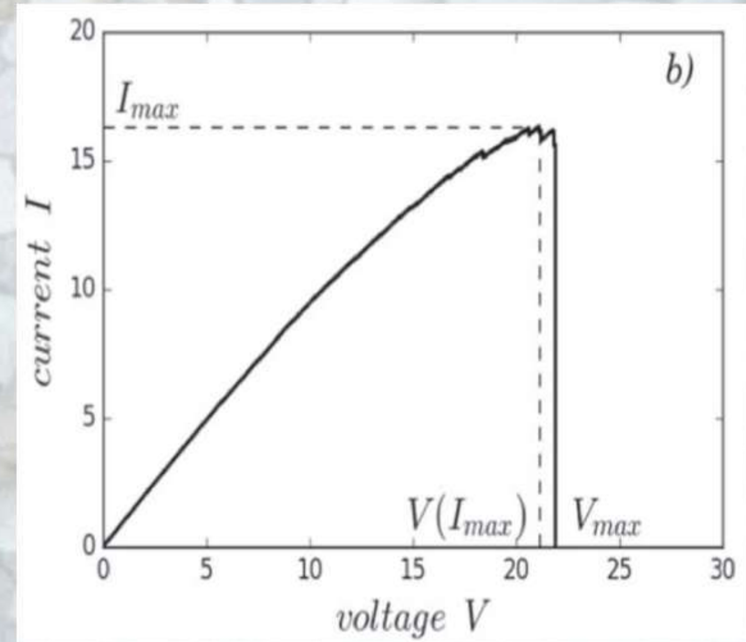
exponential decay

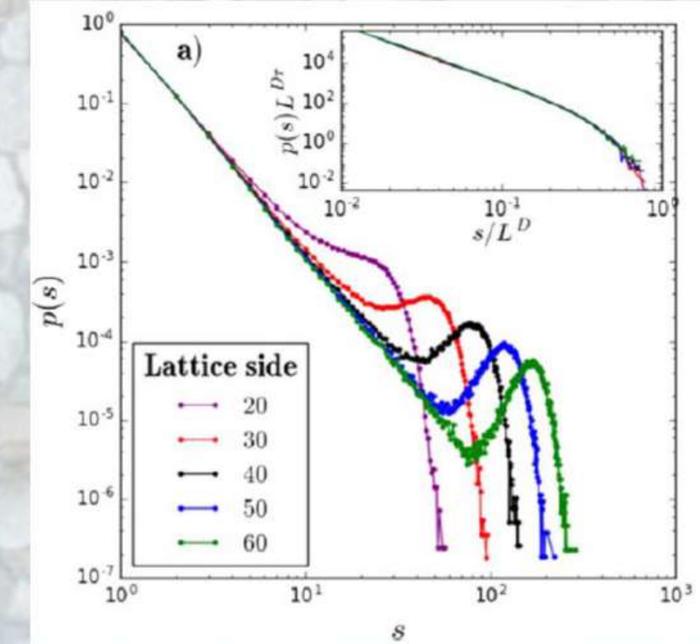
finite size scaling $\theta = 0.16 \pm 0.01$

Random fuse model

L. de Arcangelis, S. Redner, H.J.H., J. Physique Lett. 46, L585-L590 (1985)







avalanche size distribution

Diffusion on percolation clusters

at p_c

$$t \sim R^{d_w}$$

$$d_w > 2$$

$$\langle r^2(t) \rangle \sim t^{2/d_w}$$

Nernst—Einstein equation

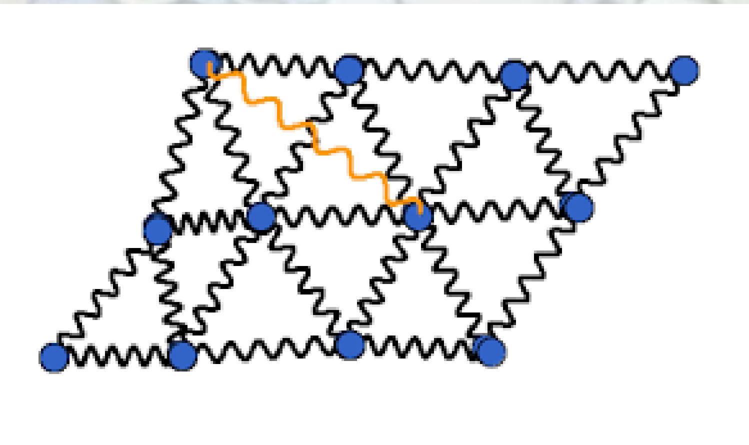
$$\sigma_{\rm dc} = n(e^2/k_B T)D$$

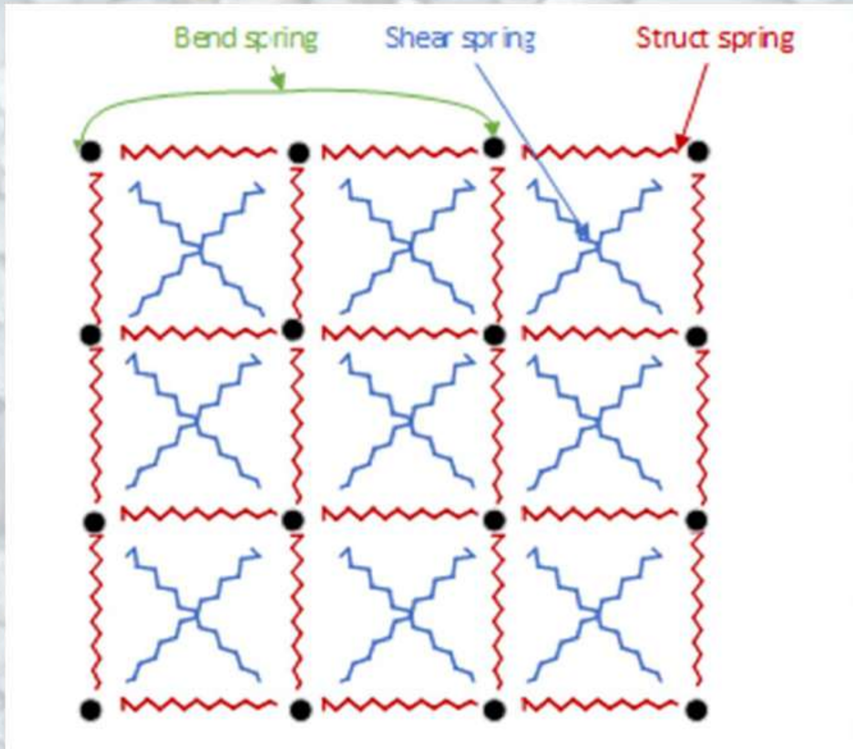
$$\longrightarrow \quad d_w = 2 + \frac{t-\beta}{\nu}$$

Rigidity percolation

Elastic behaviour of disordered solids

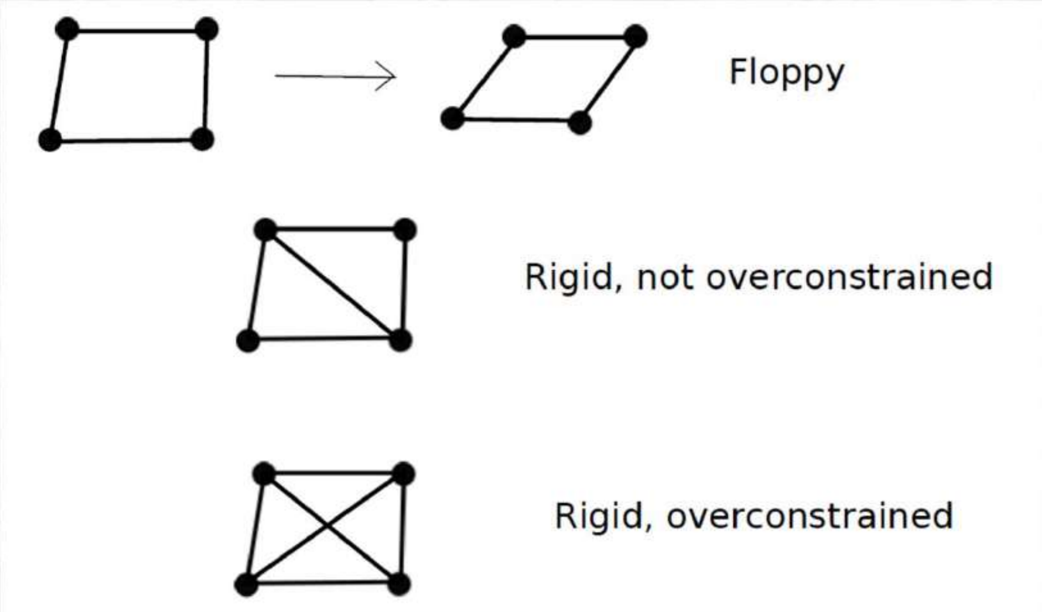
Occupy bonds with elastic springs.

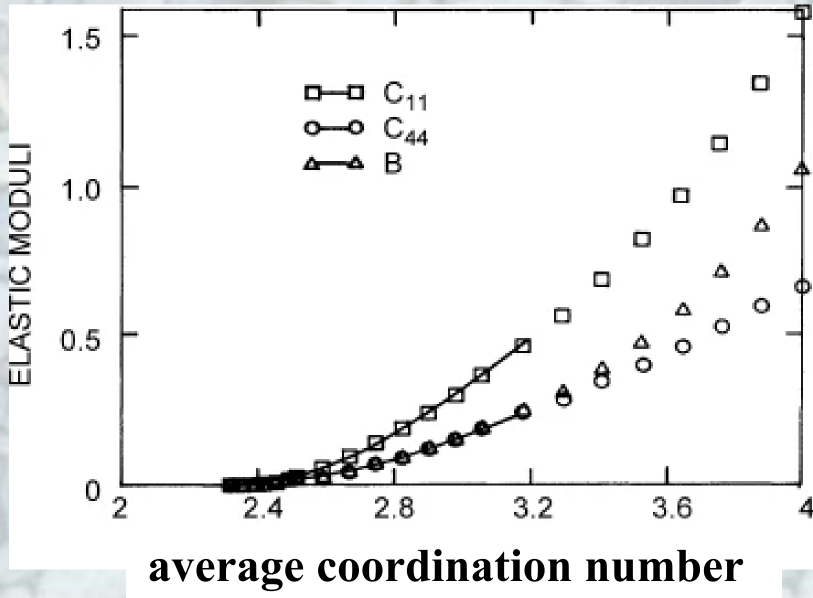




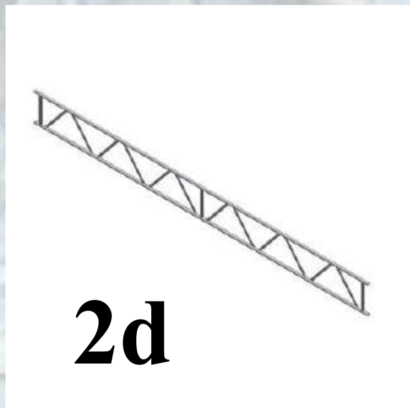
At which dilution does the solid collapse = shear modulus *G* vanishes

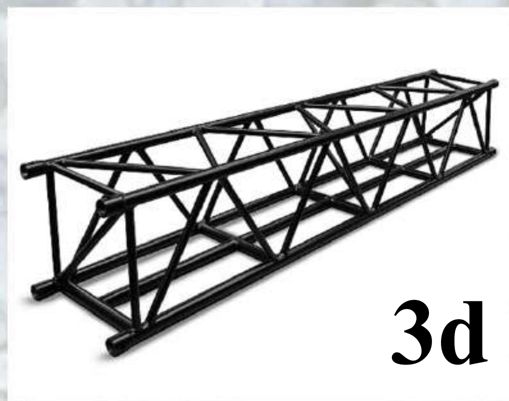
Rigidity percolation



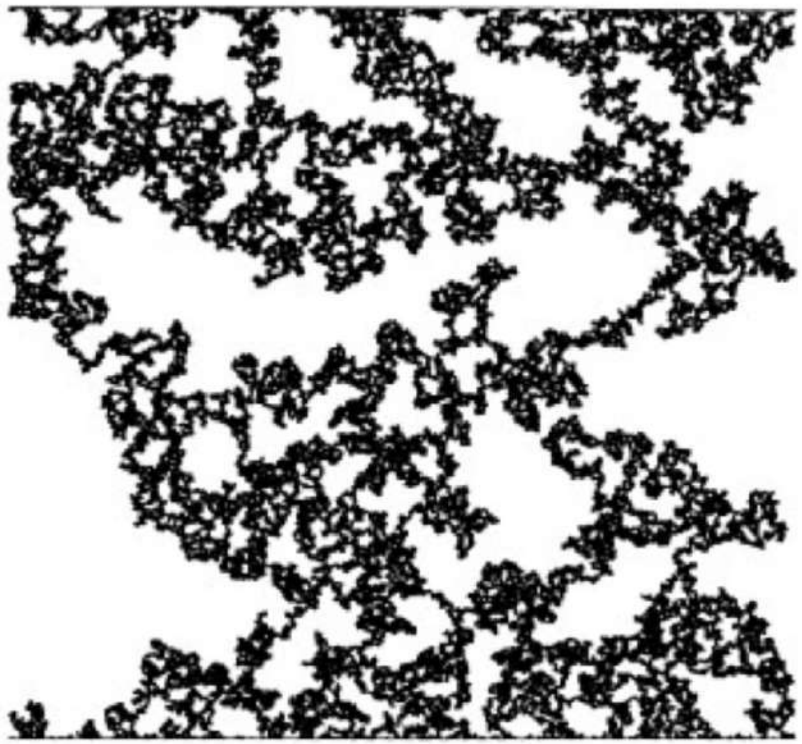


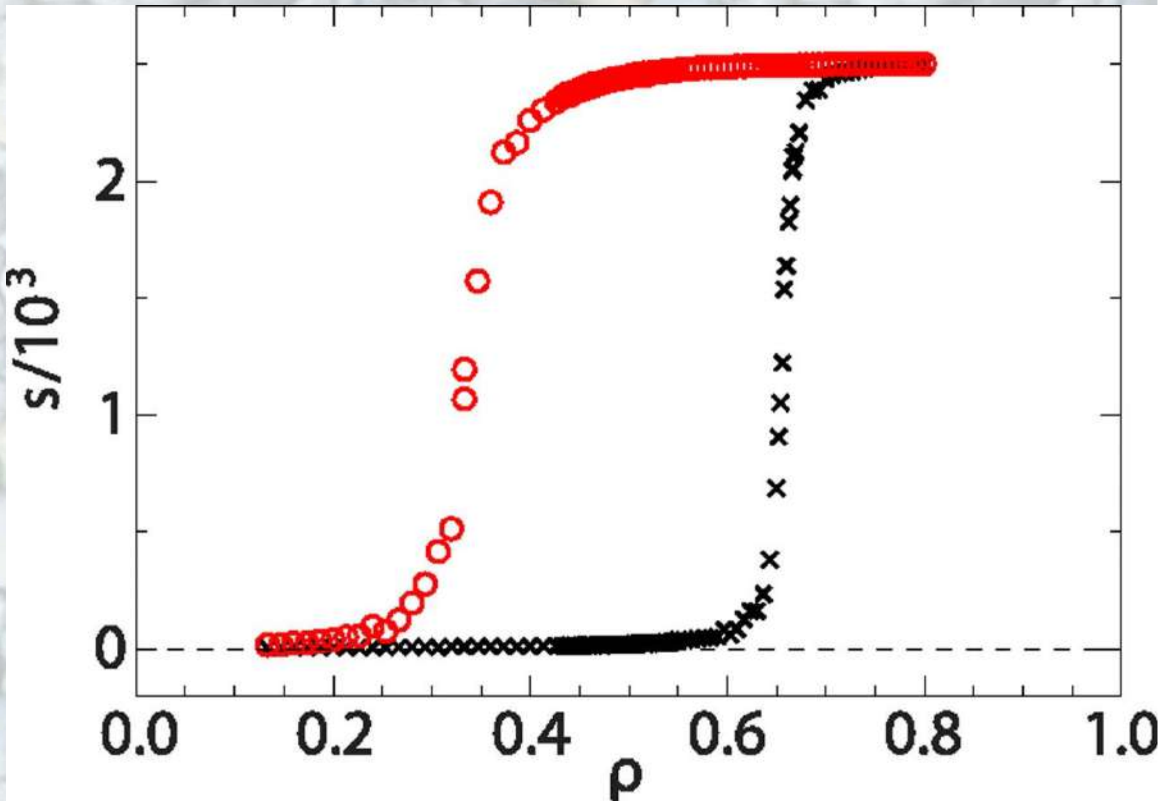
One must reinforce angles and construct trusses.





Rigidity percolation





$d_f \approx 1.95$

p_c is shifted: 0.347 -> 0.66

shear modulus

triangular lattice

$\nu \approx 1.1$

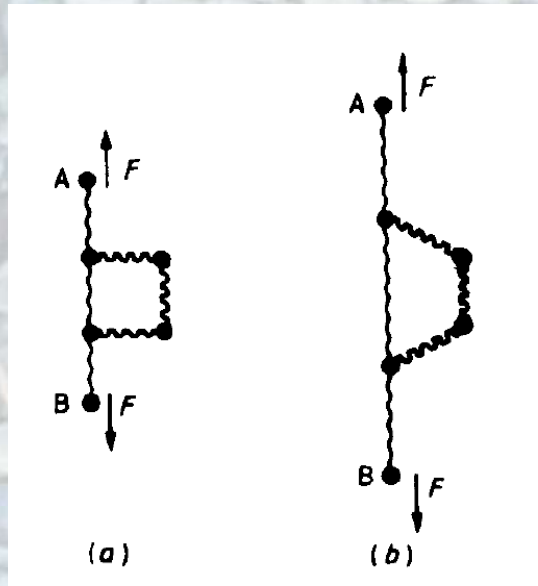
$$G \sim (p - p_{ce})^{f_c} \quad f_c \simeq 1.45$$ in 2d

Elastic Backbone

HJ Herrmann, DC Hong and HE Stanley J.Phys. A 17, L261 (1984)

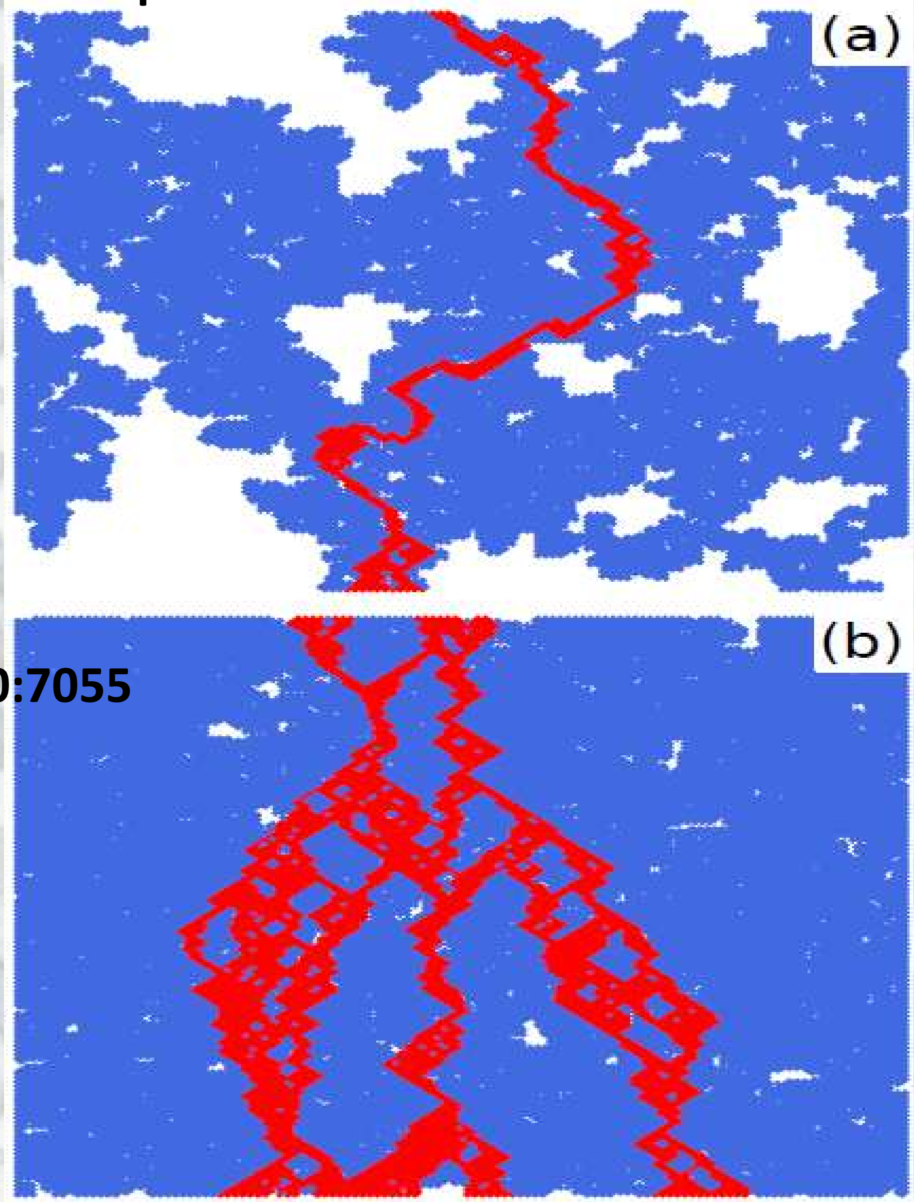
The elastic backbone is the union of all shortest paths connecting two points P_1 and P_2

It describes the elastic response of a floppy spring network.



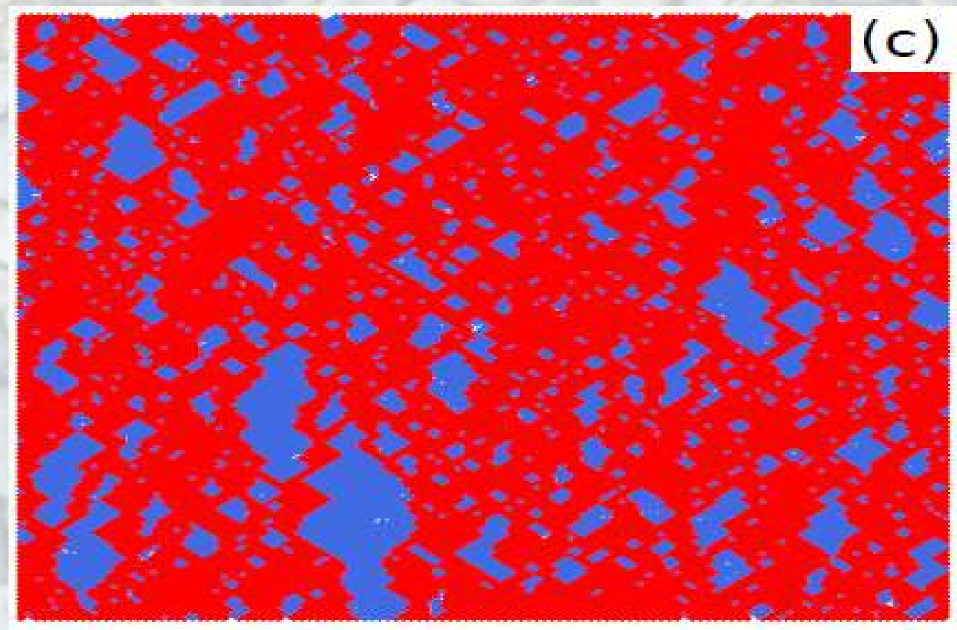
Elastic Backbone

p = 0:60



p = 0:7055

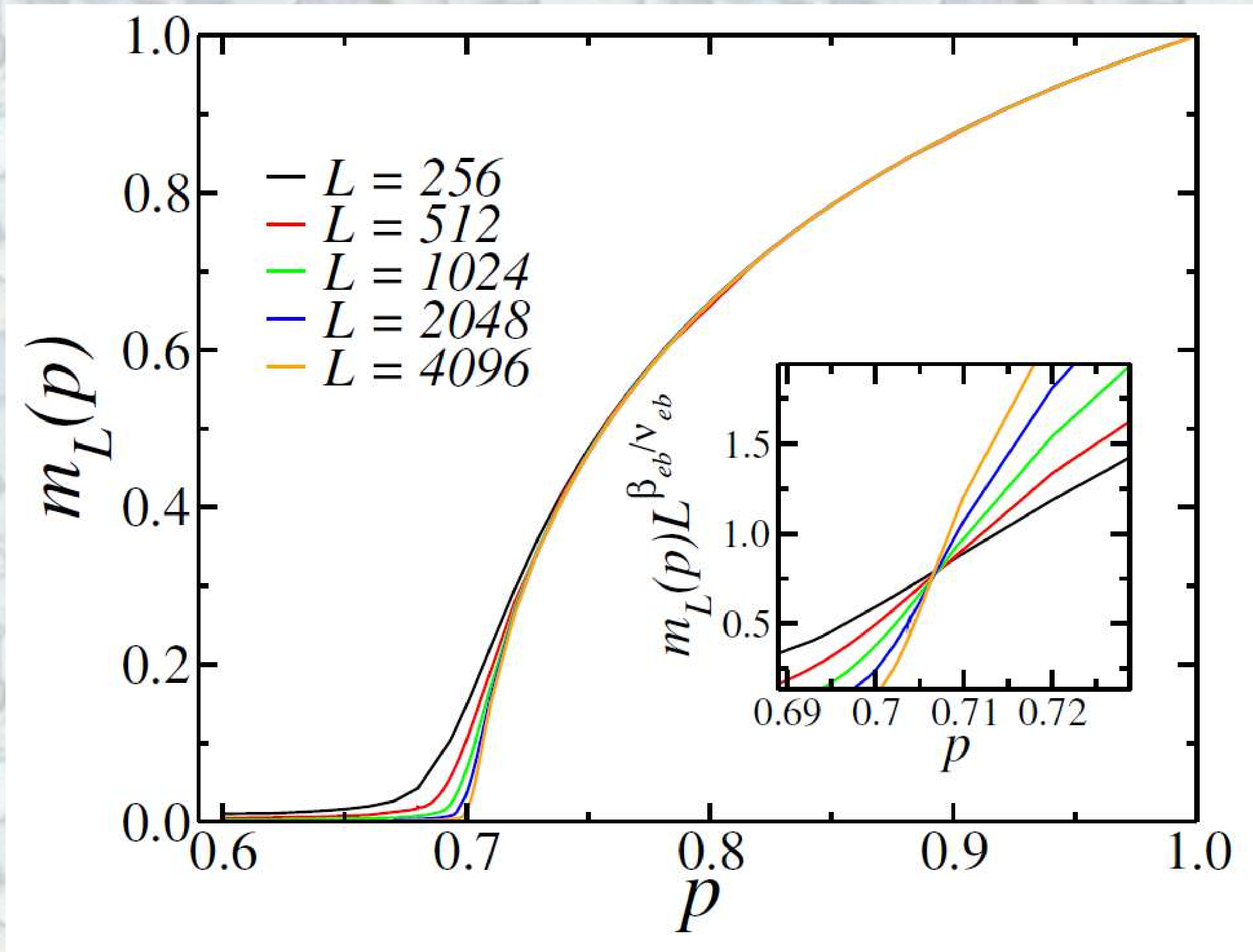
site percolation on tilted square lattice



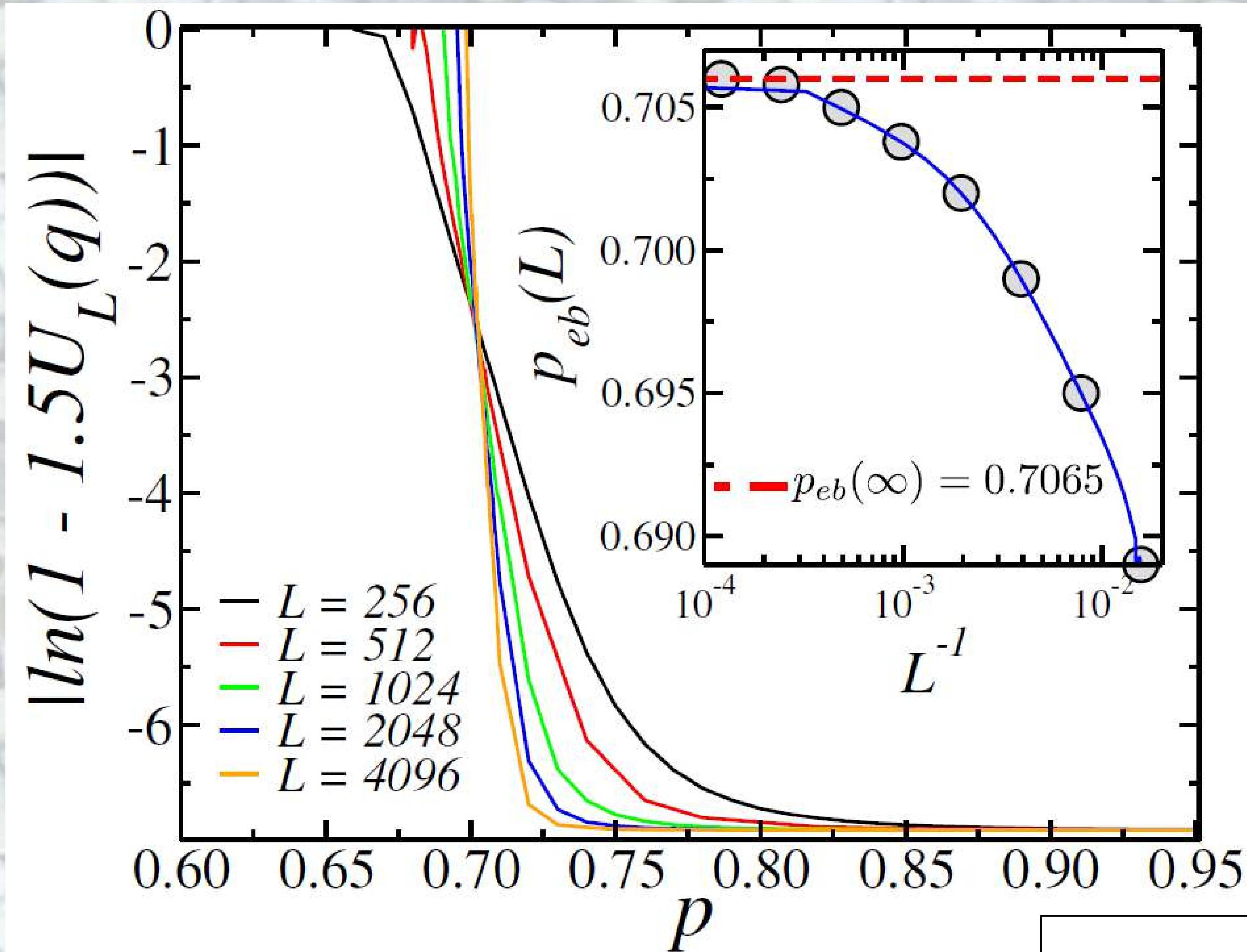
p = 0:75

C.I.N. Sampaio , J.S. Andrade Jr., H.J. H., A.A. Moreira, Phys. Rev. Lett. 120, 175701 (2018)

Elastic Backbone



Elastic Backbone



site percolation

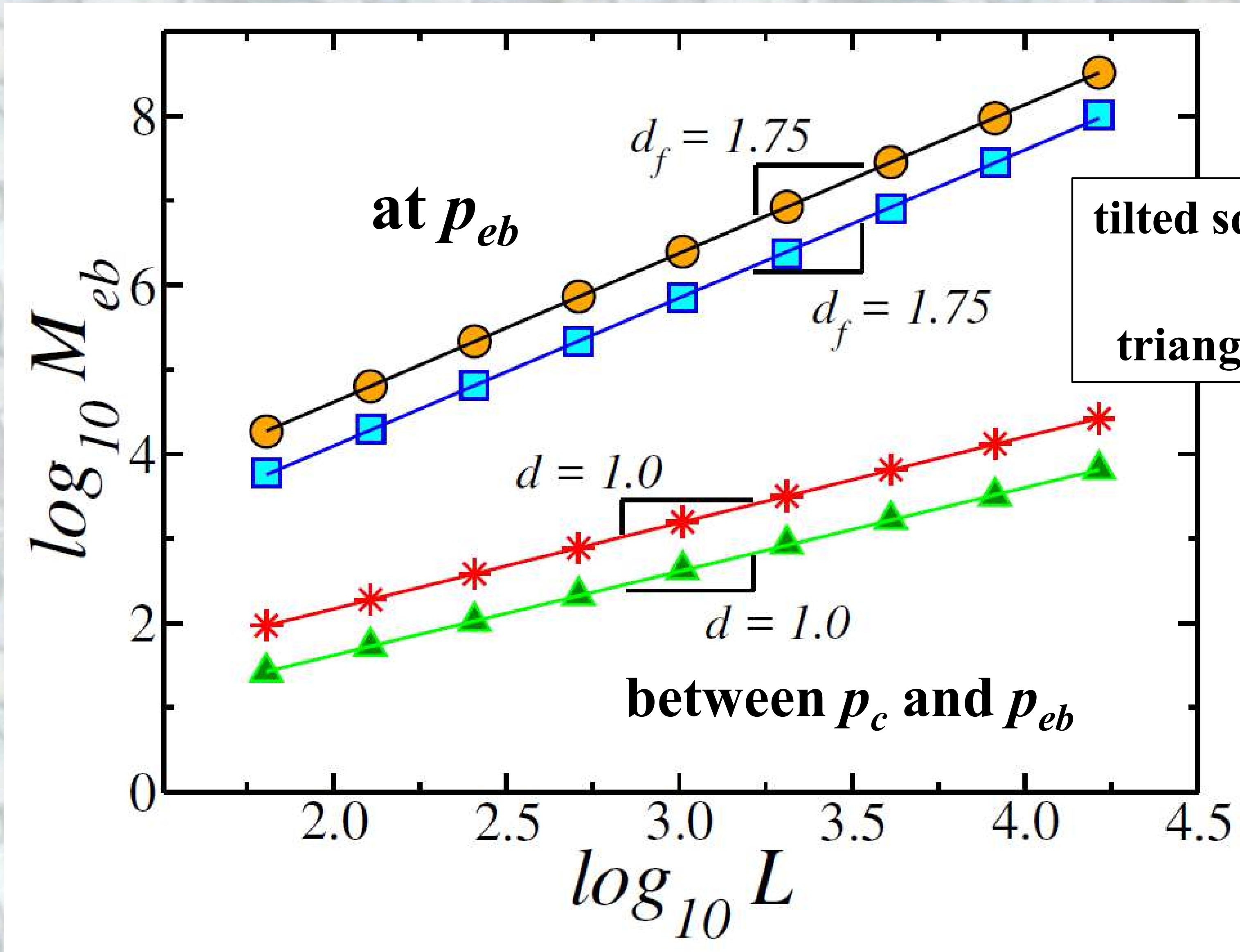
triangular lattice

Binder cumulant

$$U_L(p) = 1 - \frac{\langle m_{eb}^4 \rangle}{3 \langle m_{eb}^2 \rangle^2}$$

$$\boldsymbol{p_{eb} = 0.7065 \pm 0.0004}$$

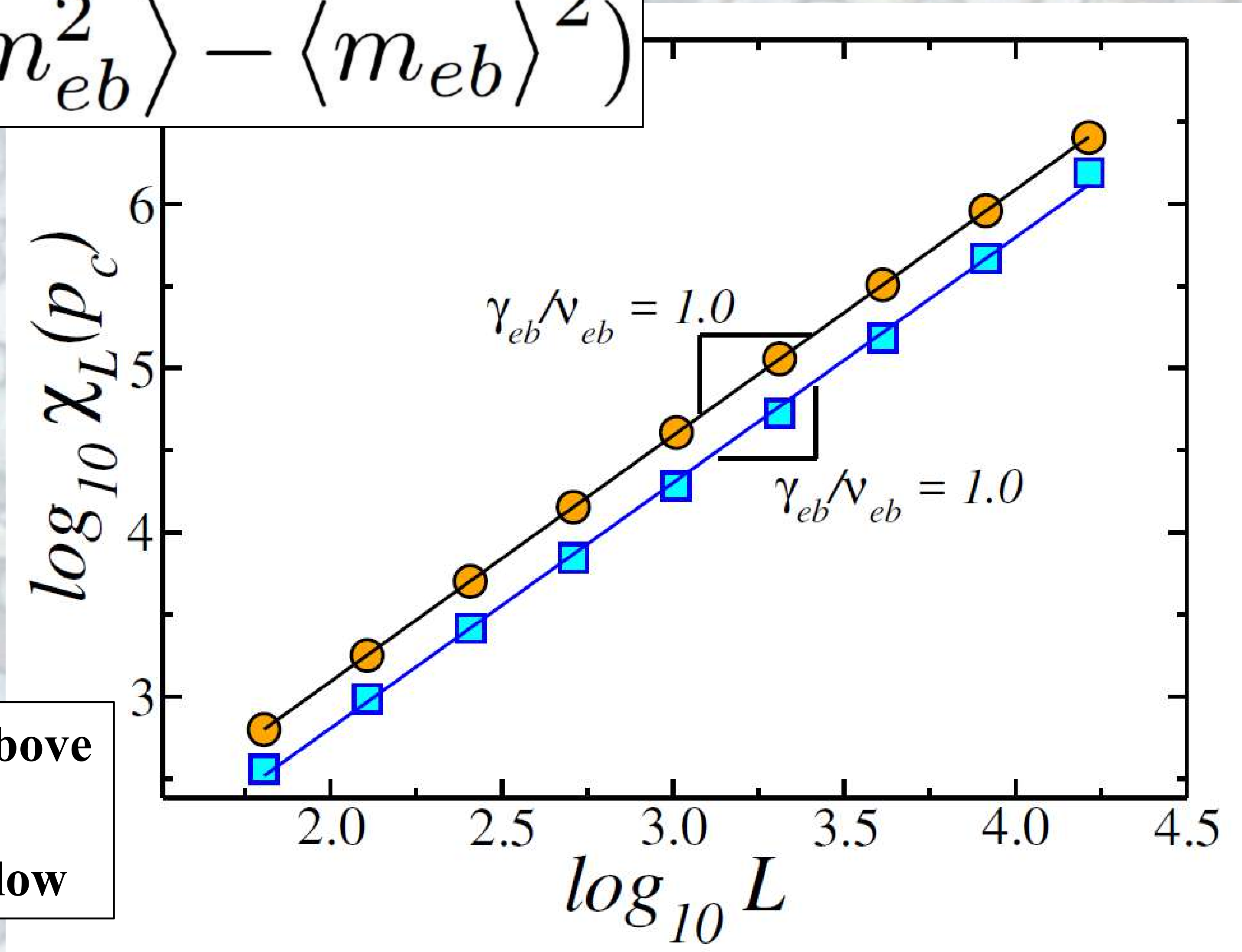
Elastic Backbone



tilted square lattice above
and
triangular lattice below

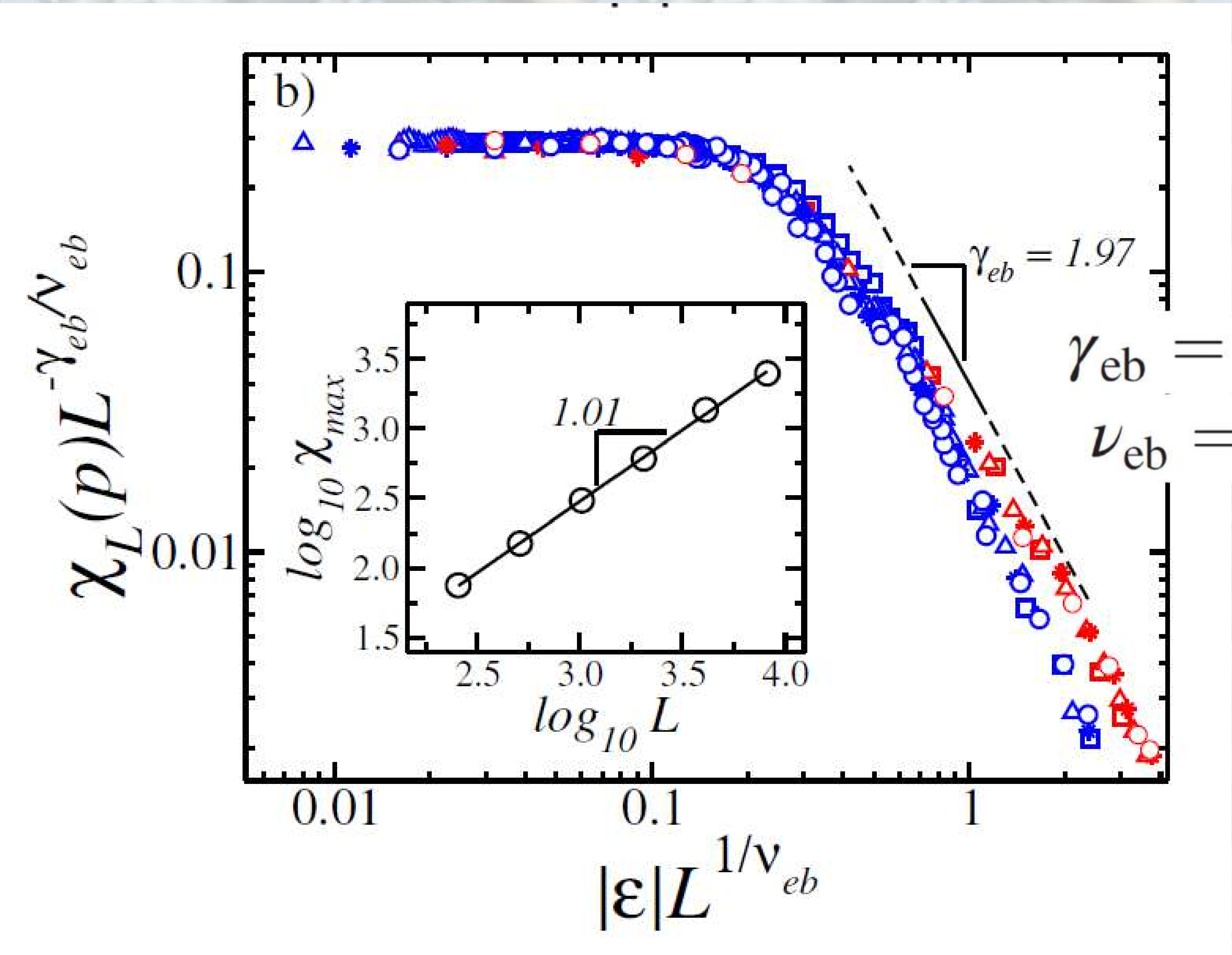
Elastic Backbone

$$\chi = N(\langle m_{eb}^2 \rangle - \langle m_{eb} \rangle^2)$$



tilted square lattice above
and
triangular lattice below

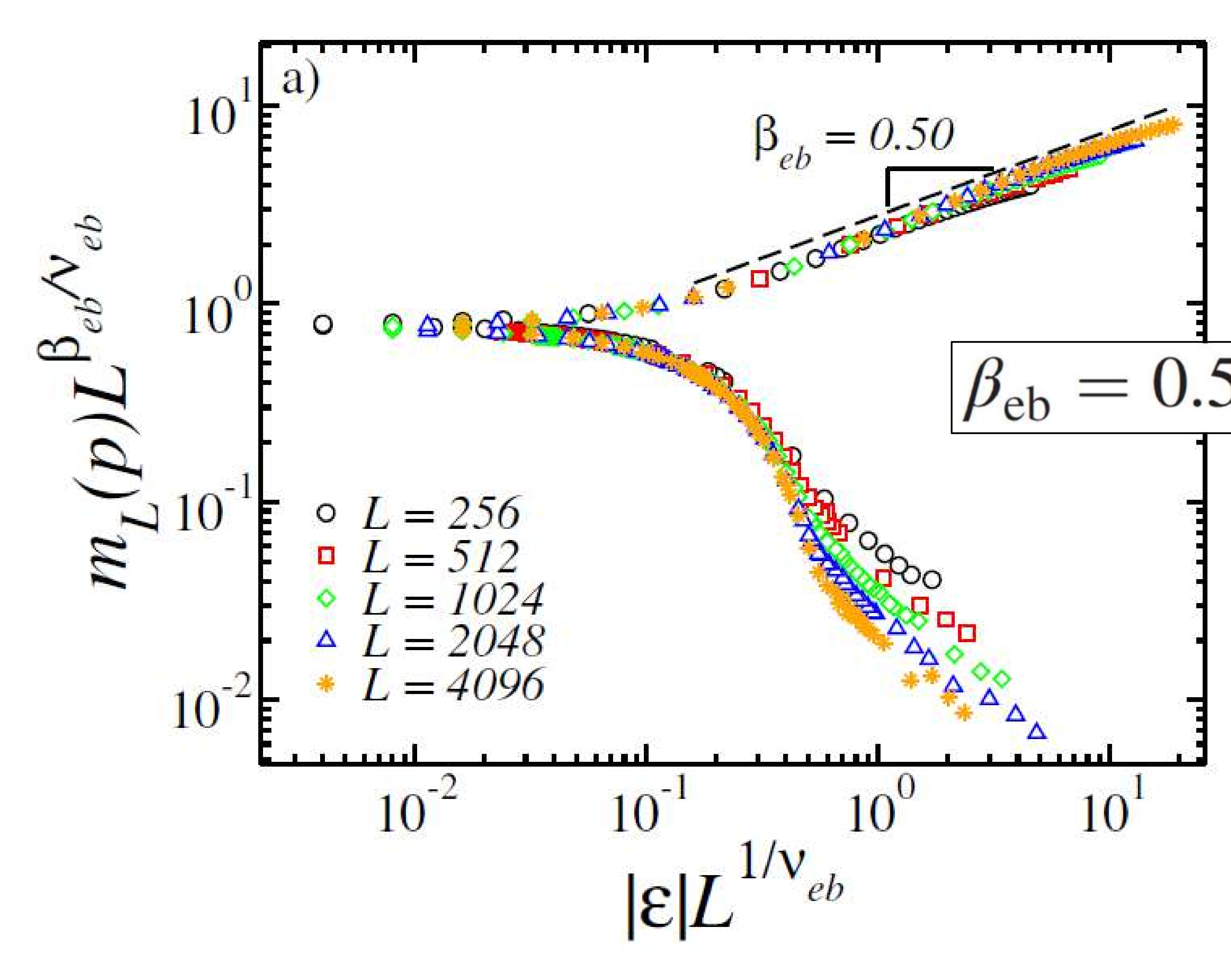
Elastic Backbone



$$\gamma_{\rm eb} = 1.97 \pm 0.05$$

$$\nu_{\rm eb} = 2.00 \pm 0.02$$

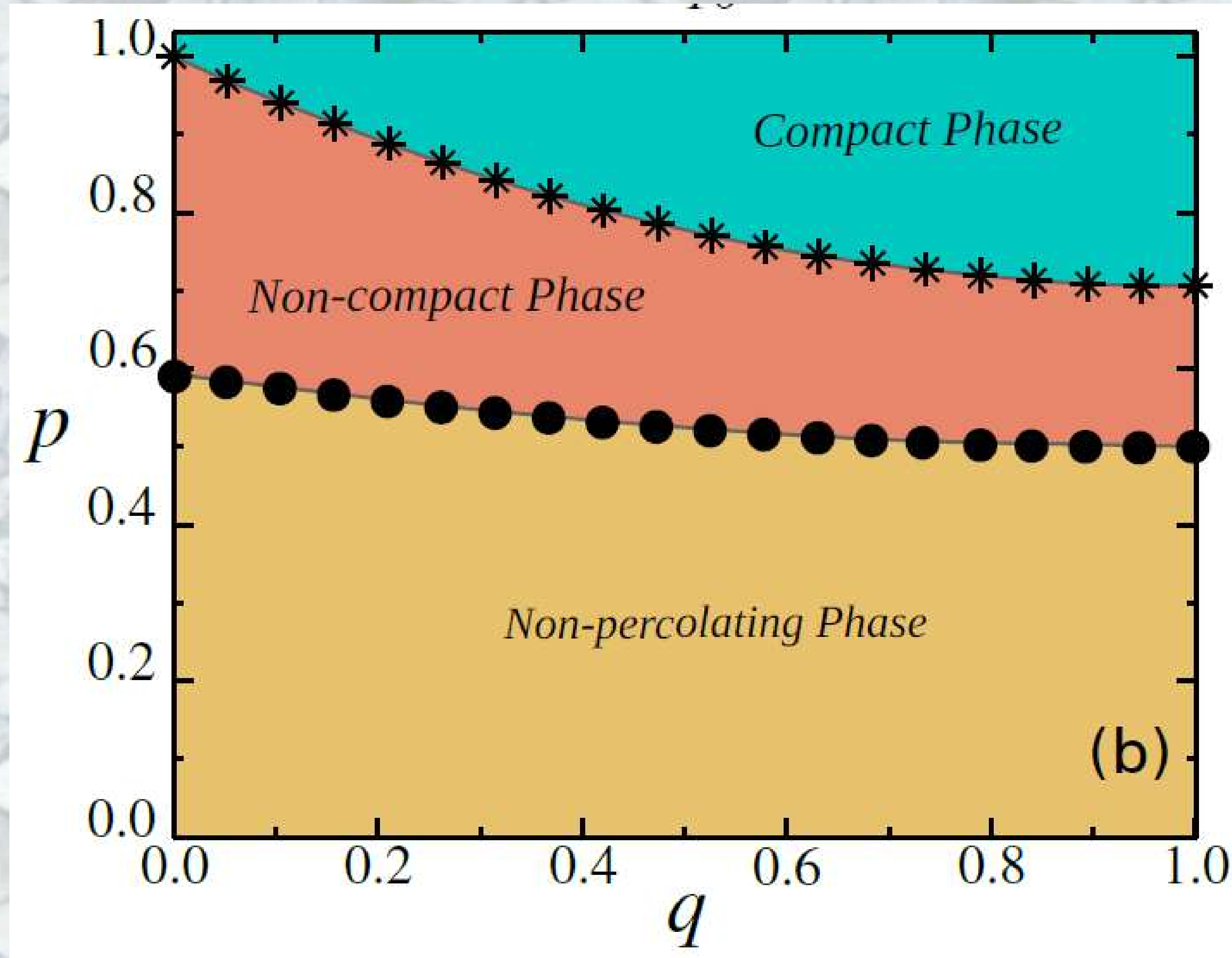
Elastic Backbone



$\beta_{\rm eb} = 0.50 \pm 0.02$

Elastic Backbone

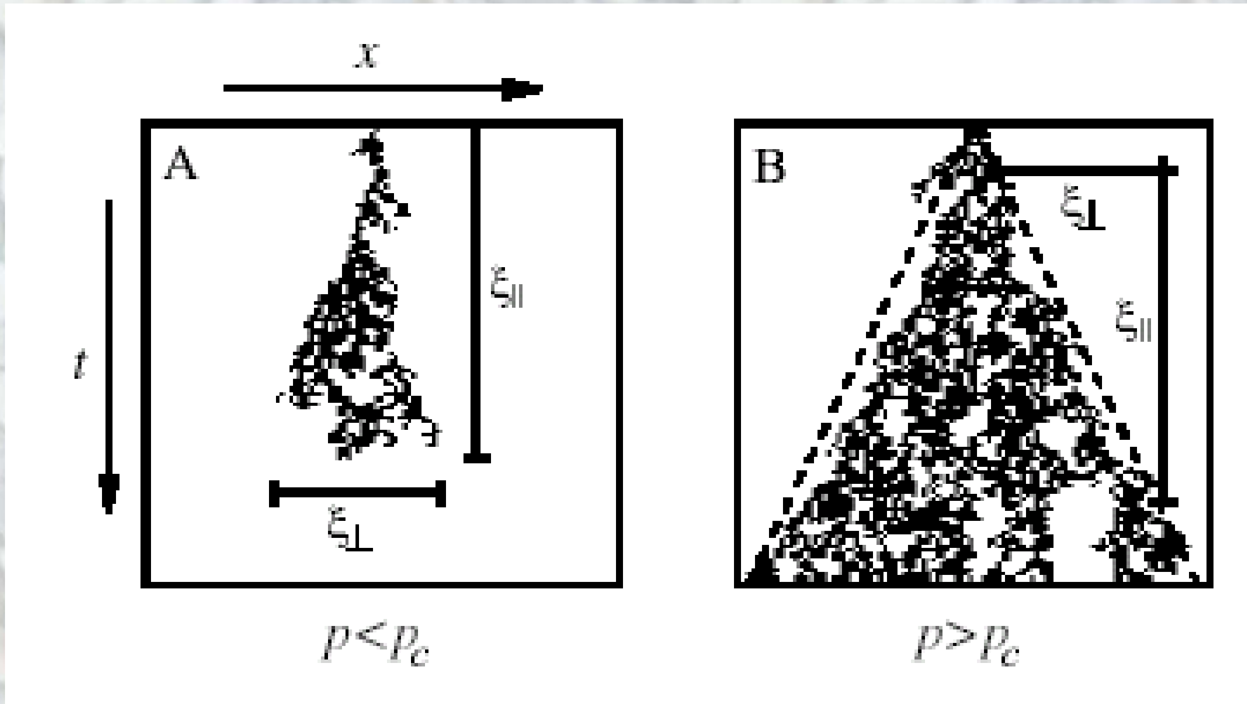
square lattice



triangular lattice

Directed percolation

= percolation on a directed lattice



If all bonds point in the same direction one can identify this direction with time t.

healing phase spreading phase

$p_c = 0.644700185(5)$ (Jensen,99)

two different correlation lengths:

$$\xi_\parallel \sim (p-p_c)^{-\nu_\parallel} \qquad \nu_\parallel = 1.73$$

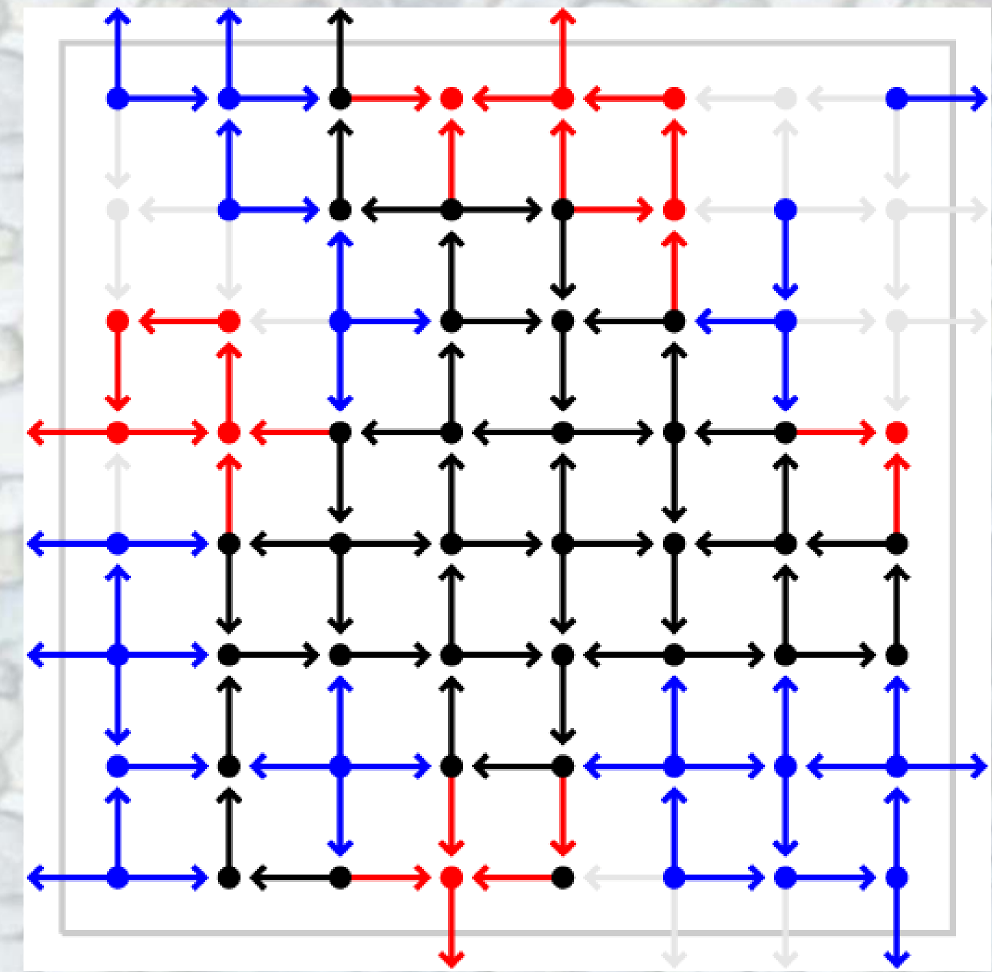
$$\xi_\perp \sim (p-p_c)^{-\nu_\perp} \qquad \nu_\perp = 1.09$$

in 1+1 dimensions

Directed percolation

randomly isotropically distributed orientation of bonds

black:
strongly connected
component
red + black:
outgoing component
blue + black:
incoming component



A.W.T. de Noronha, A.A. Moreira, A.P. Vieira, H.J.H., J.S. Andrade, H.A. Carmona, Phys. Rev. E 98, 062116 (2018)

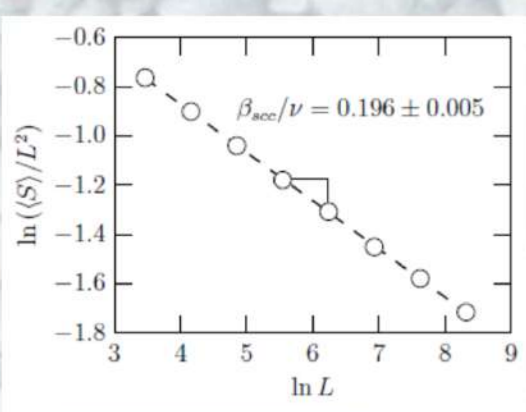
Directed percolation

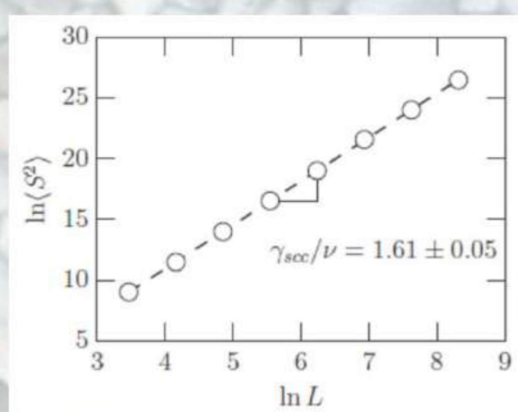
Two types of clusters can be defined:

1. strongly connected ones are the sets of points that can be mutually reached following strictly the bond directions

2. directionally connected ones are all the sites that can be reached from a given site following bond directions

Directionally connected clusters are in the universality class of standard percolation, while strongly connected clusters have different exponents.

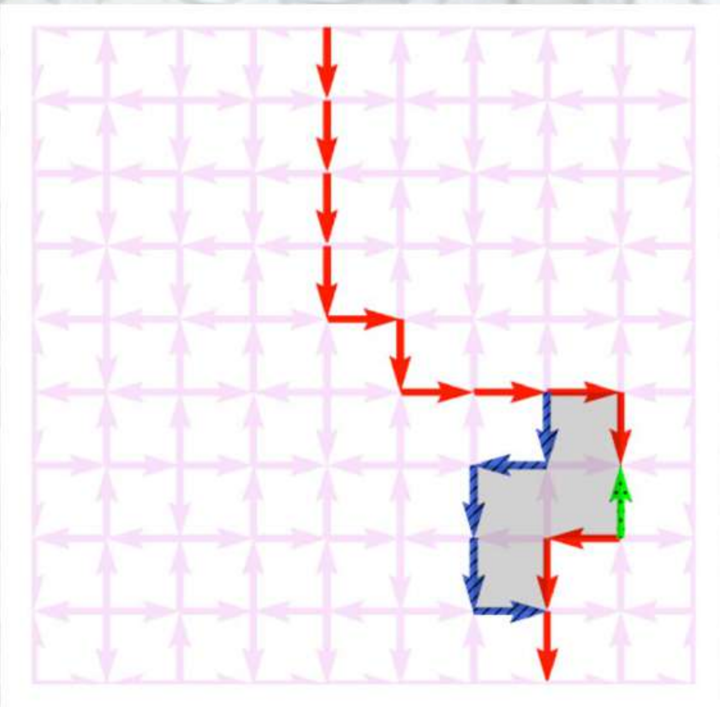


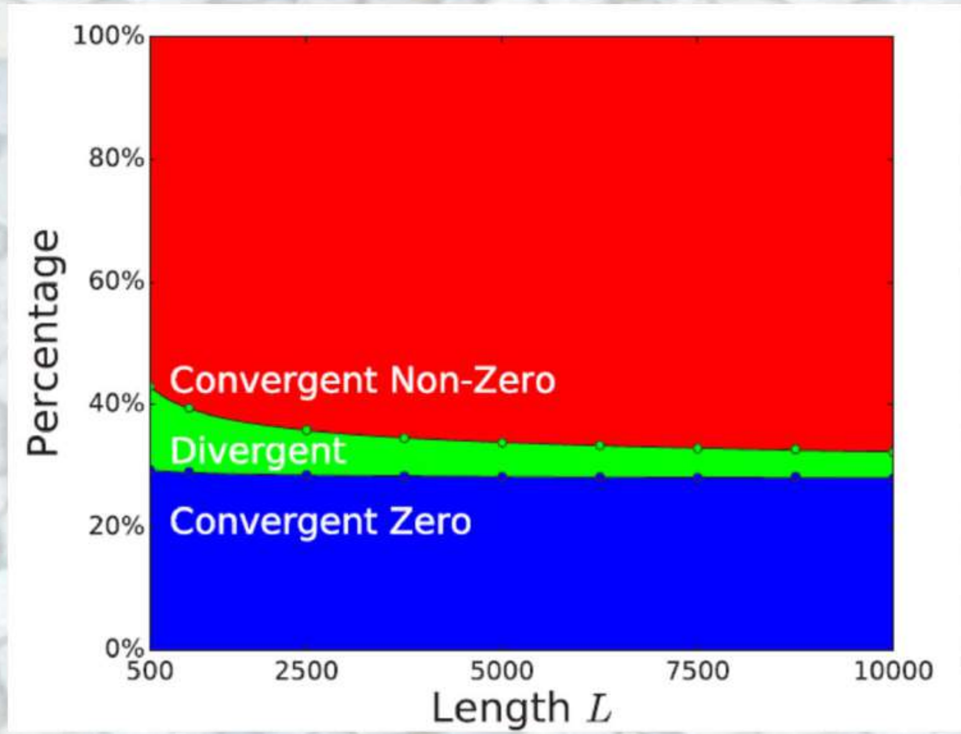


can be realized experimentally with electric diodes

Disturbing the shortest path of directed percolation

Consider the shortest path on a square lattice with randomly distributed orientation of bonds. Then, flip the direction of one single bond along the path.

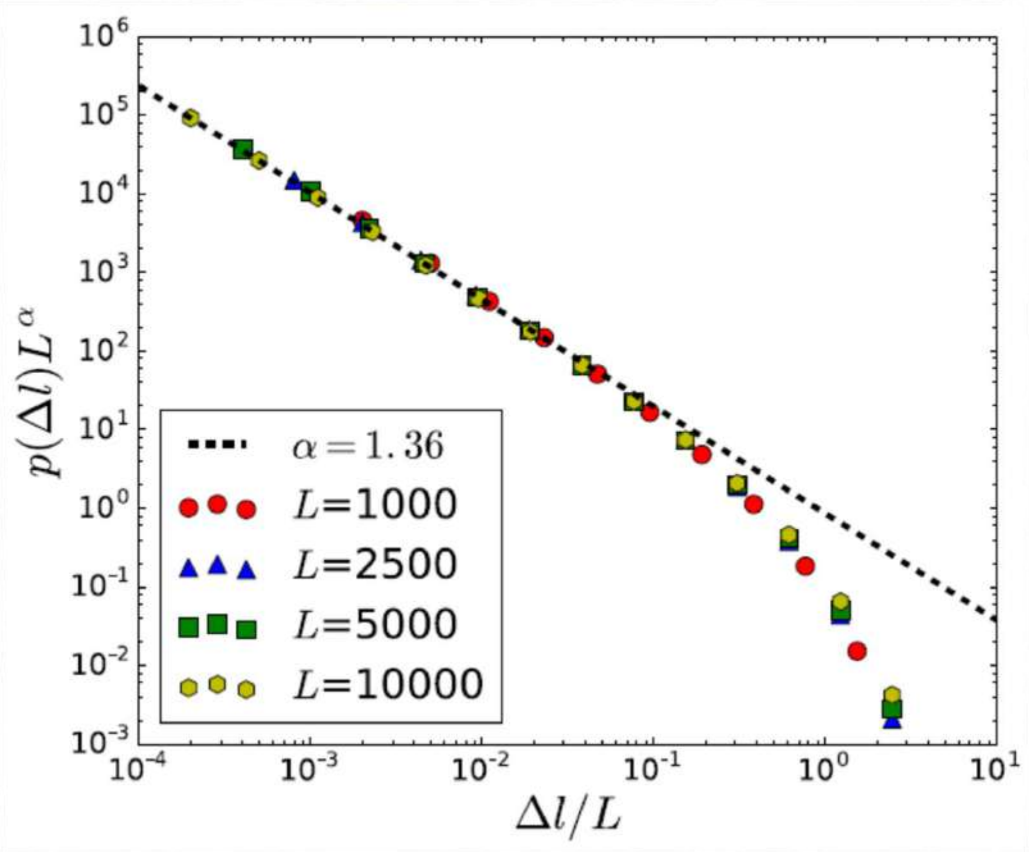




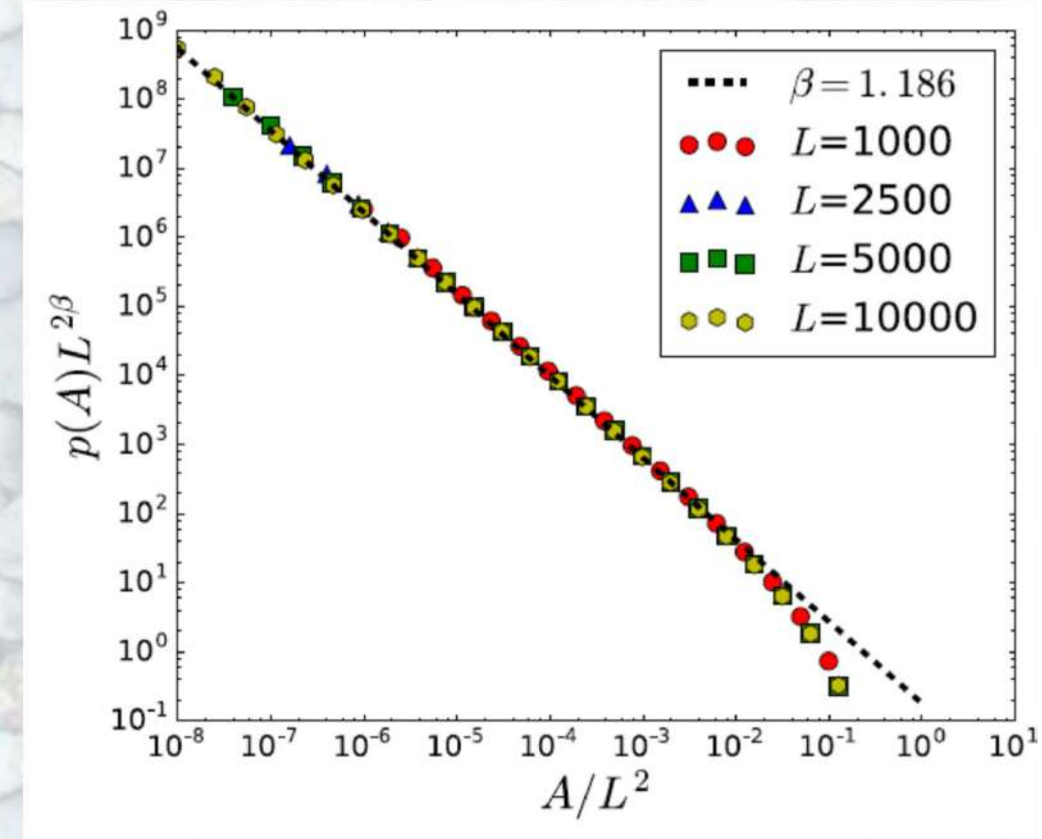
F. Hillebrand, M. Lukovic, H.J.H., Phys. Rev. E 98, 052143 (2018)

Disturbing the shortest path in directed percolation

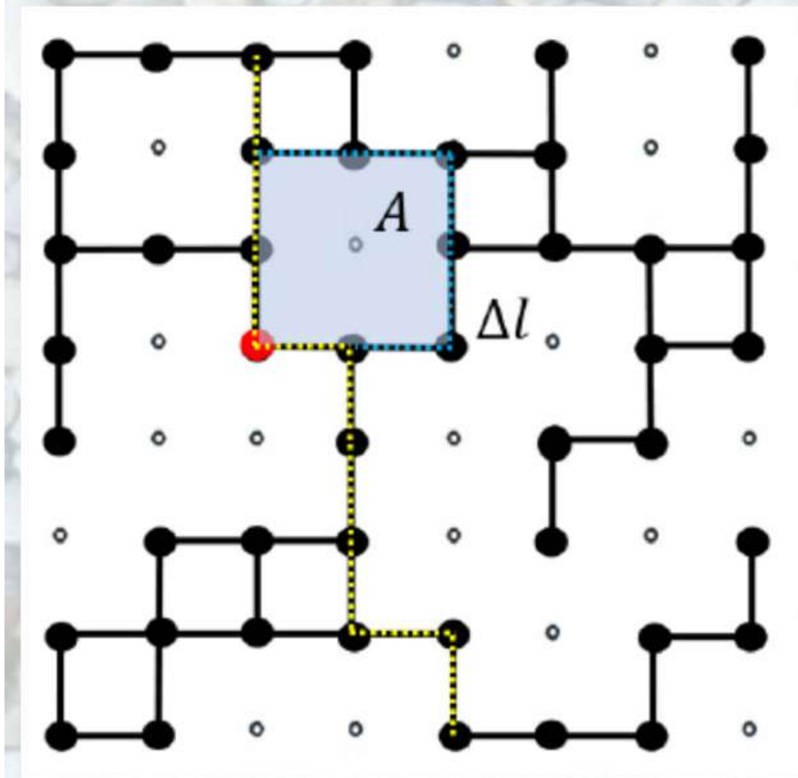
distribution of differences in path lengths



distribution of enclosed areas



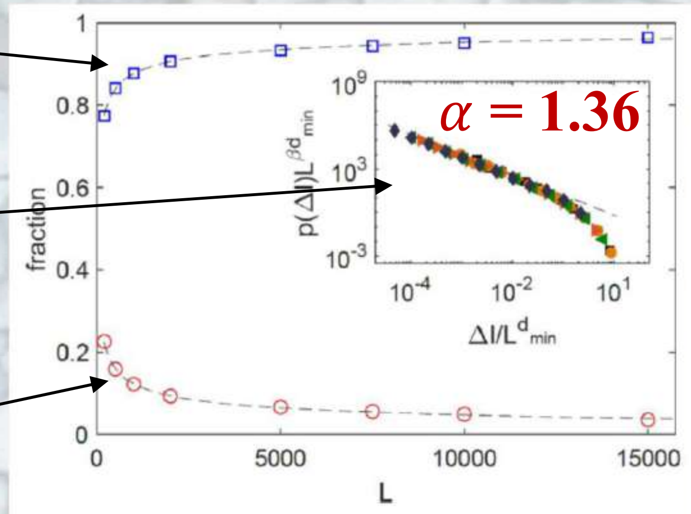
Sequential disruption of the shortest path in isotropic percolation



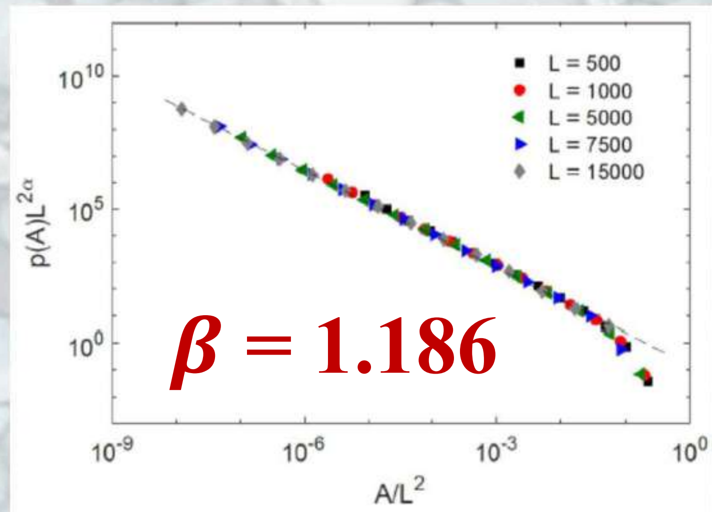
converging paths

distribution of
length differences
for converging paths

diverging paths

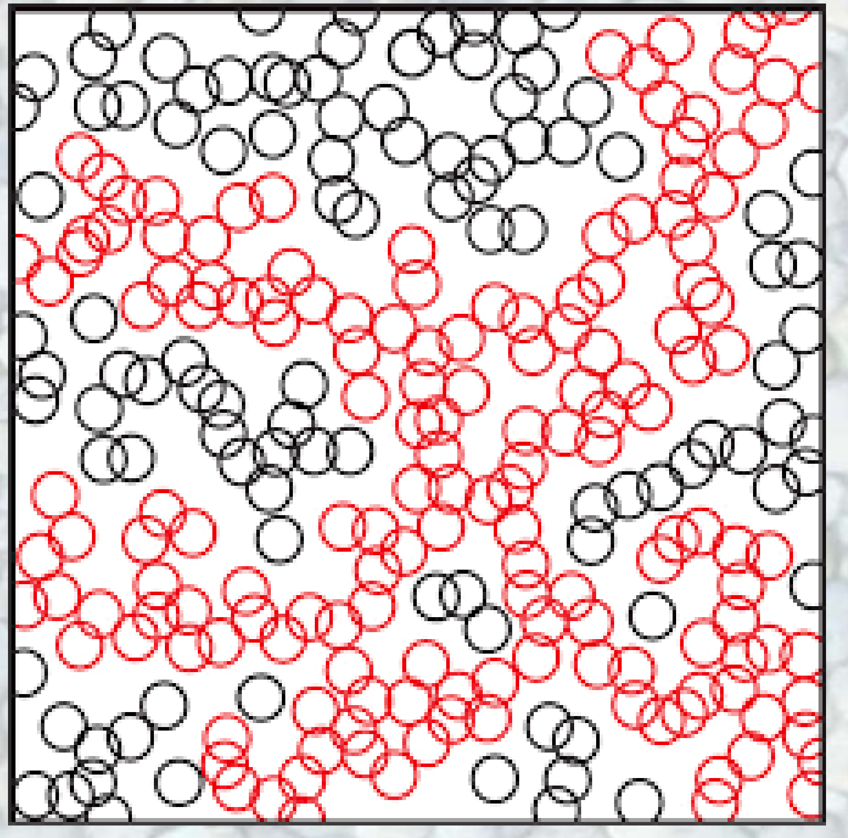


distribution of areas
for converging paths

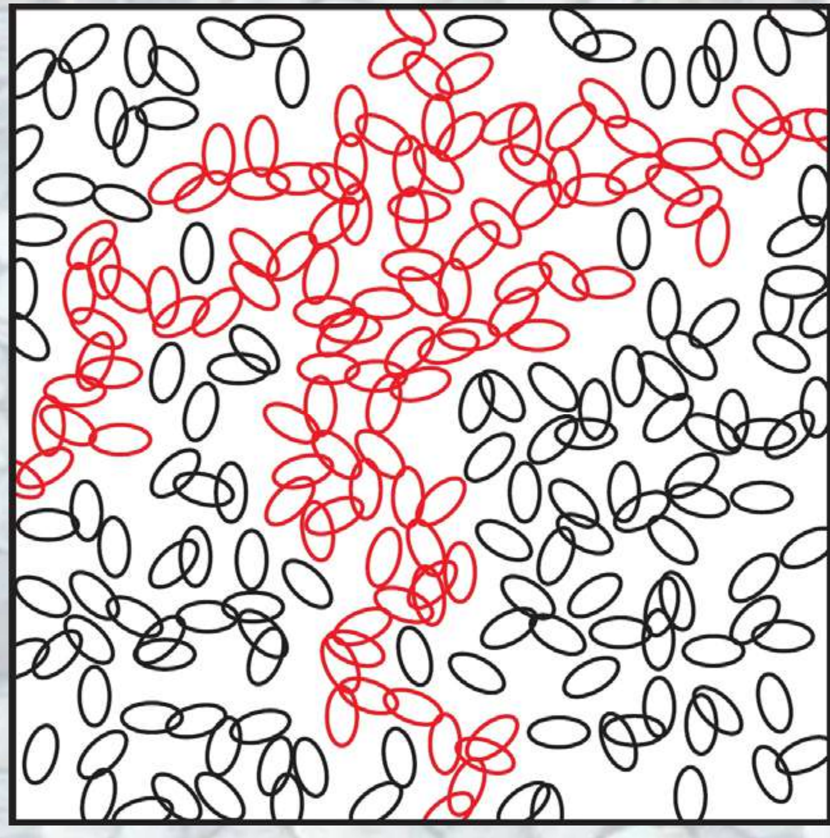


O. Gschwend, H.J.H., Phys.Rev. E 100, 032121 (2019)

Continuum percolation



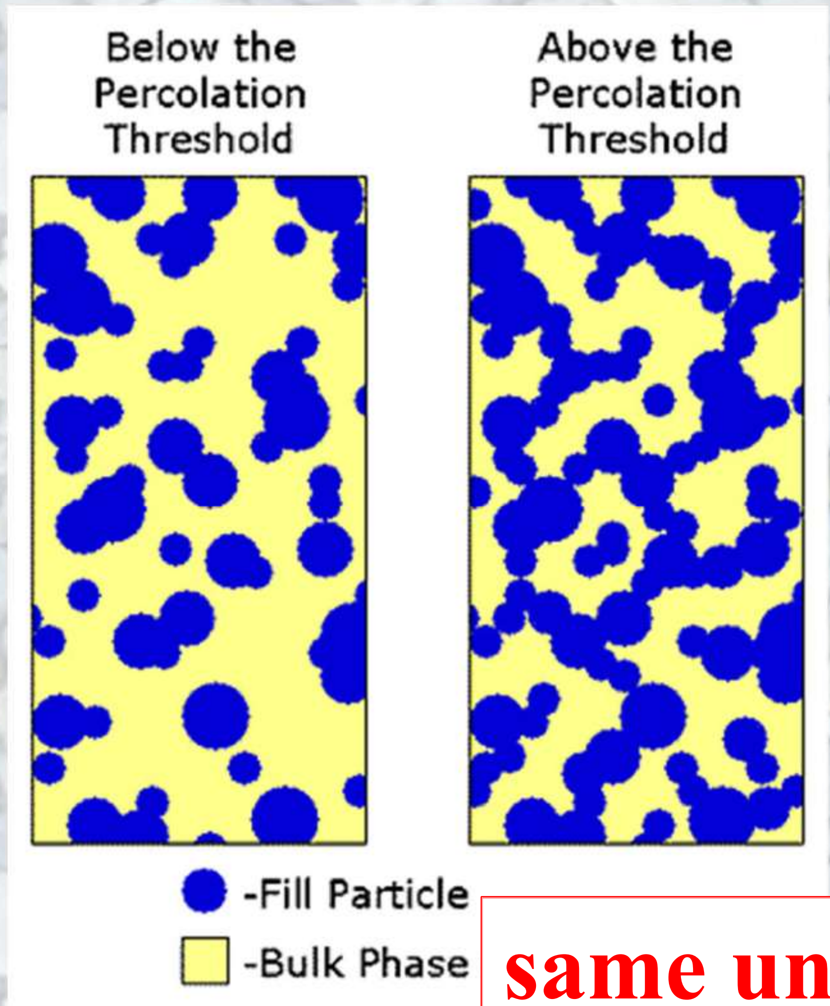
$\phi_c \approx 0.676$

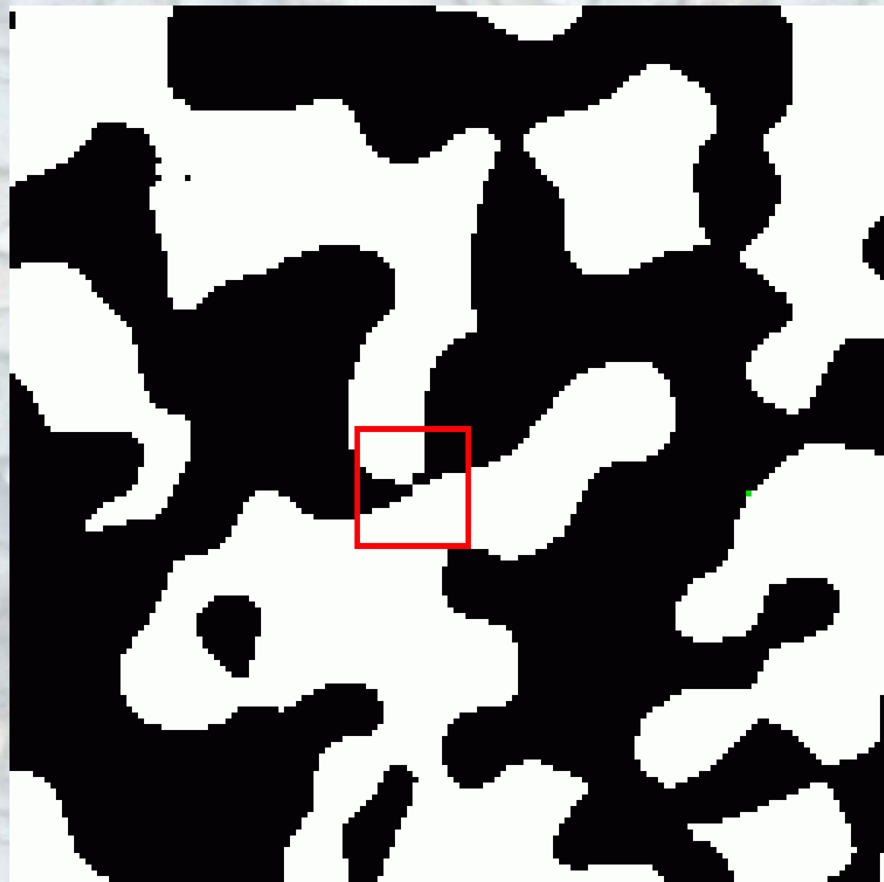


$\phi_c \approx 0.628$

Continuum percolation

Swiss cheese model = void model

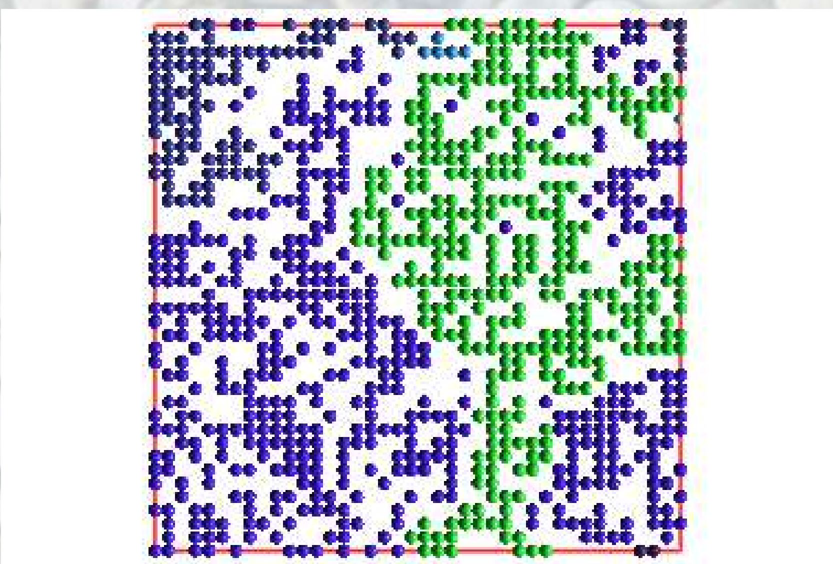
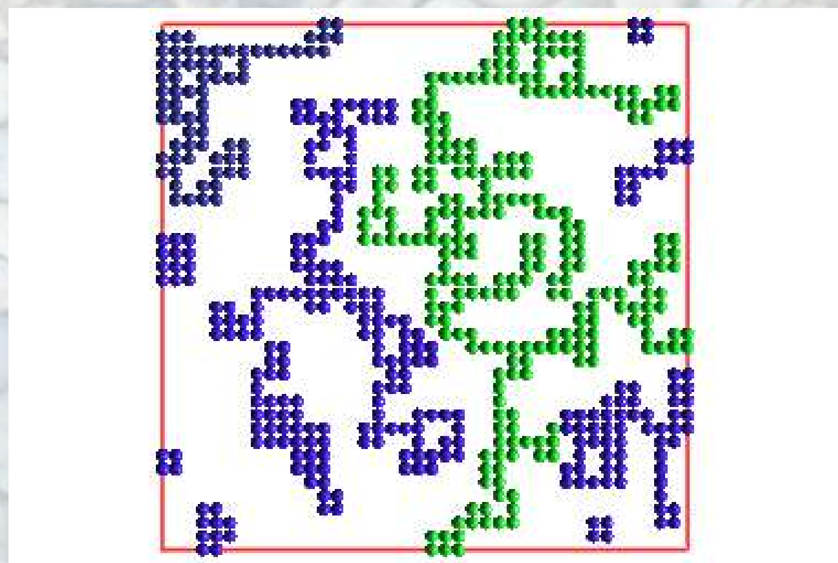




same universality class as on lattice

Bootstrap percolation

Chalupa, Leath and Reich (1979)

Start with $p = 0.55$ on square lattice.

Remove iteratively all sites that have less than $m = 2$ occupied neighbors: „culling“.

Bootstrap percolation

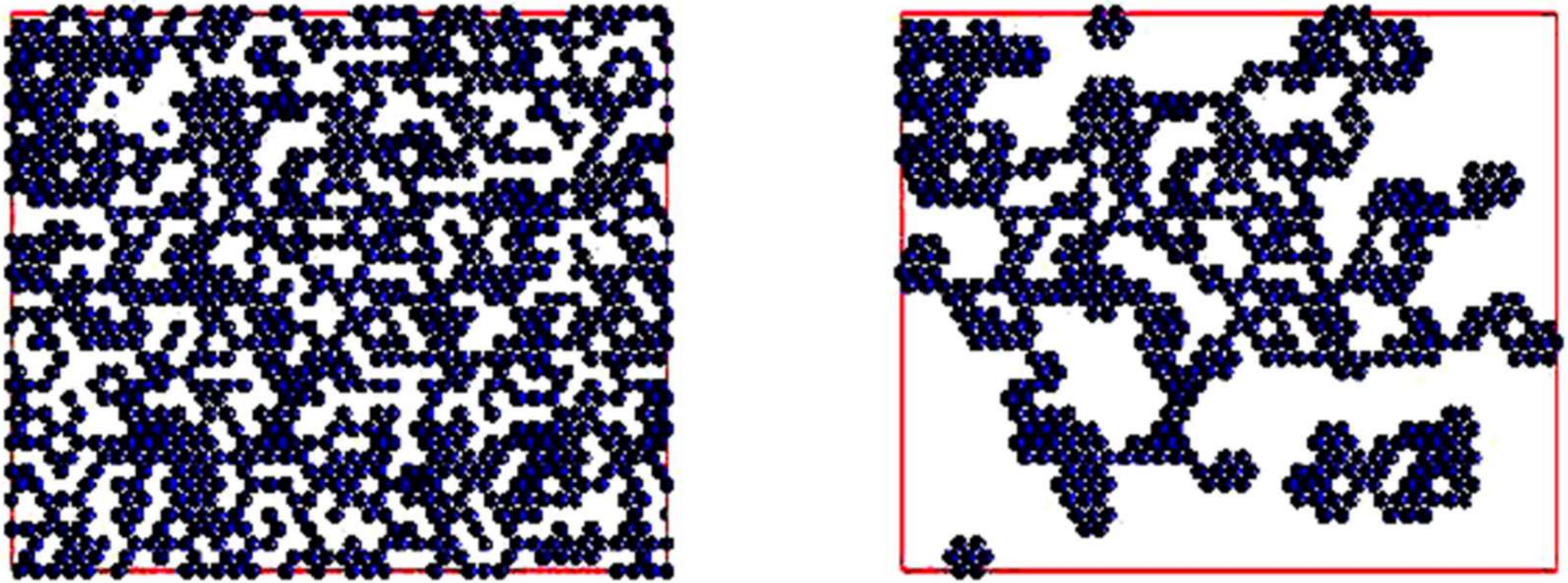
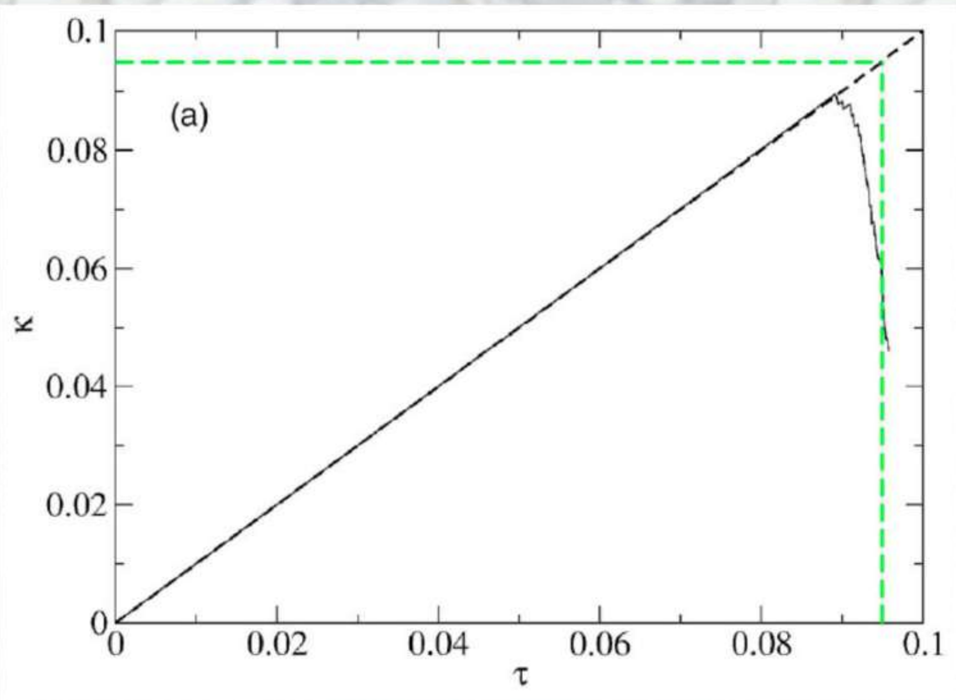


Figure 2. The initial freshly occupied lattice shown on the left for $m = 3$ on the triangular lattice at an initial concentration of $p = 0.66$, above the usual percolation threshold of $p_c = 1/2$ for this lattice. For initial occupation there is indeed an infinite cluster, but after culling there is a more compact cluster that does not percolate, as shown on the right.

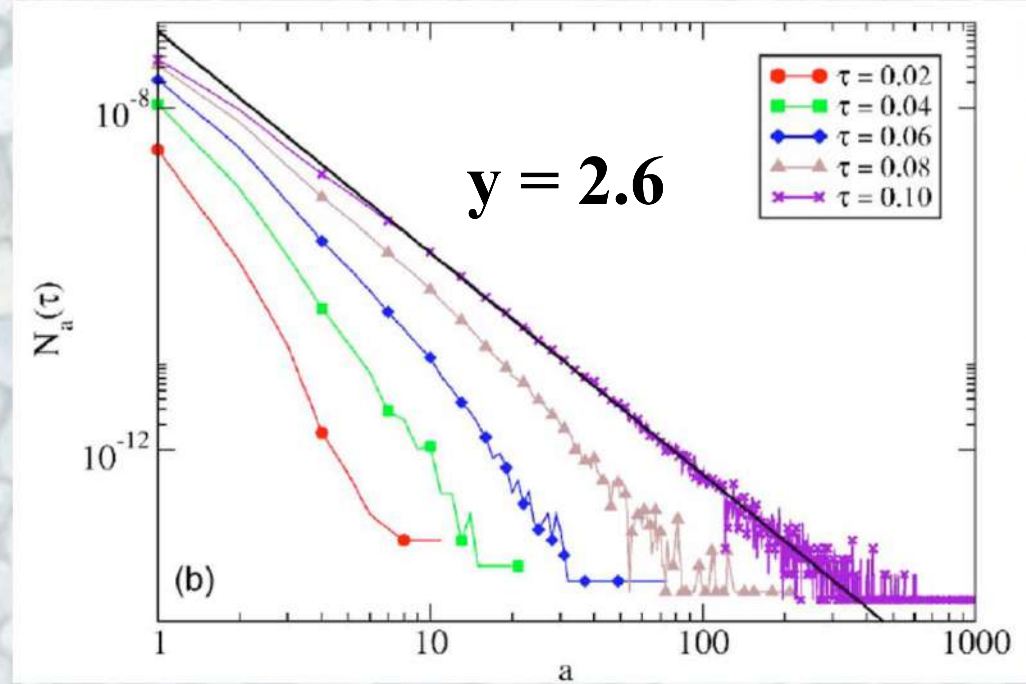
triangular lattice, *m* = 3

Bootstrap percolation

evolution of number of removed sites



culling avalanche size distribution

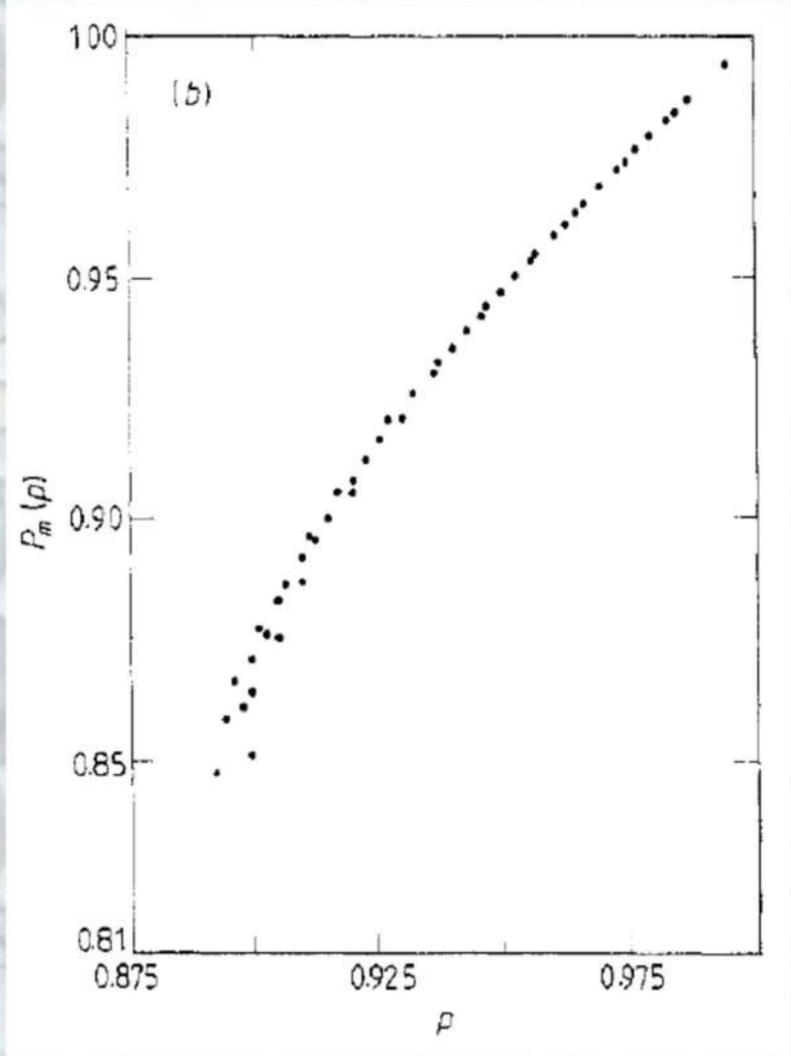


triangular lattice, *m* = 4

Farrow, Duxbury and Moukarzel (2008)

Bootstrap percolation

discontinuous (first order) transition

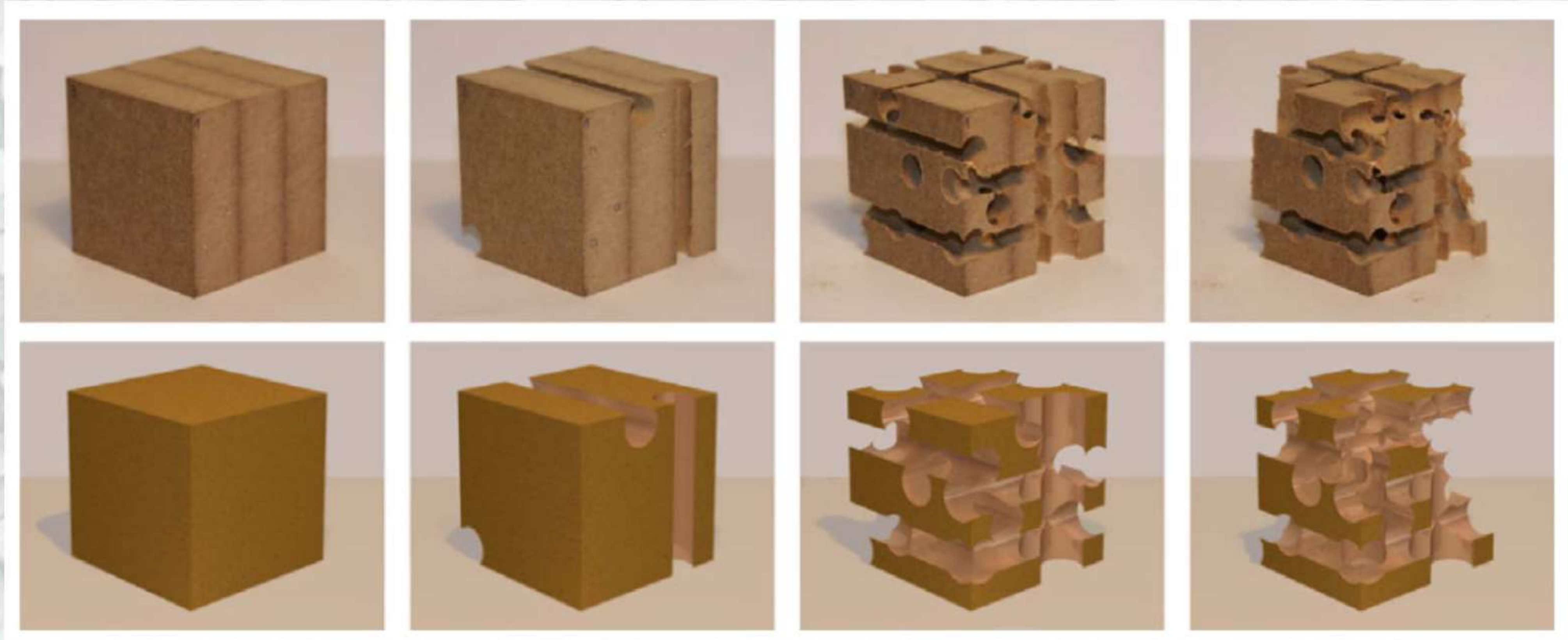


cubic lattice, *m* = 4

P M Kogut and P L Leath, *J. Phys. C* **14** 3187 (1981)

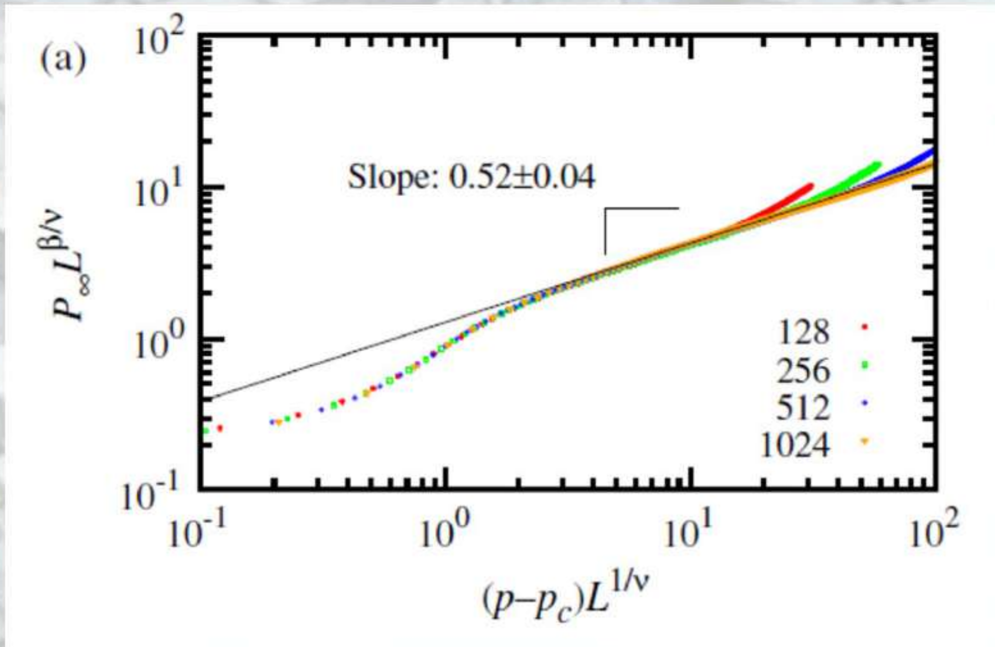
Drilling percolation

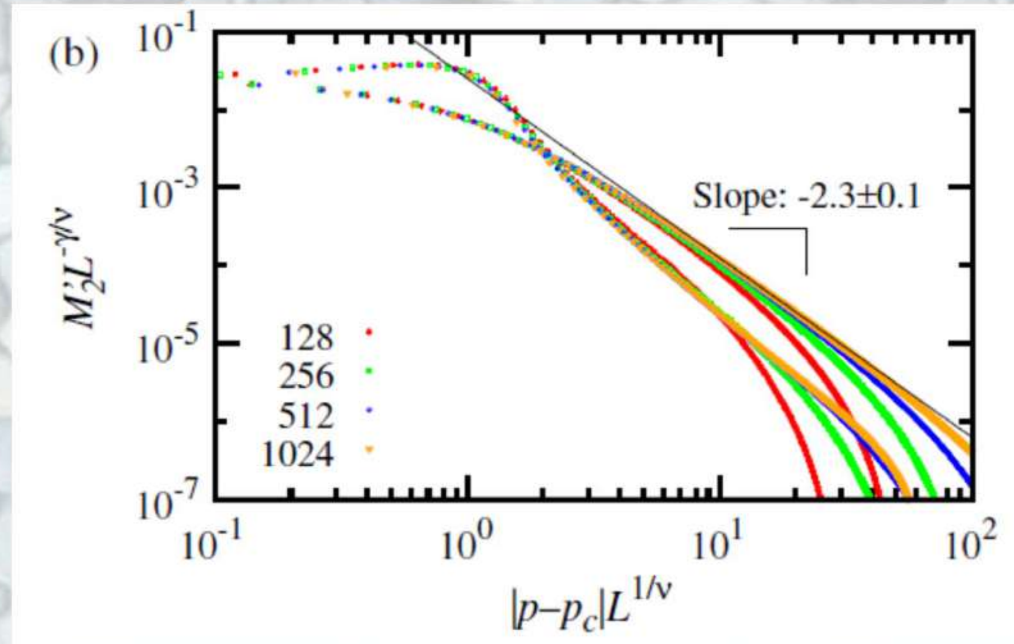
Drill in each direction (1-*p*)L^2 holes.



K.J. Schrenk, M.R. Hilário, V. Sidoravicius, N.A.M. Araújo, H.J.H., M. Thielmann, A. Texeira, Phys. Rev. Lett. 116, 055701 (2016)

Drilling percolation



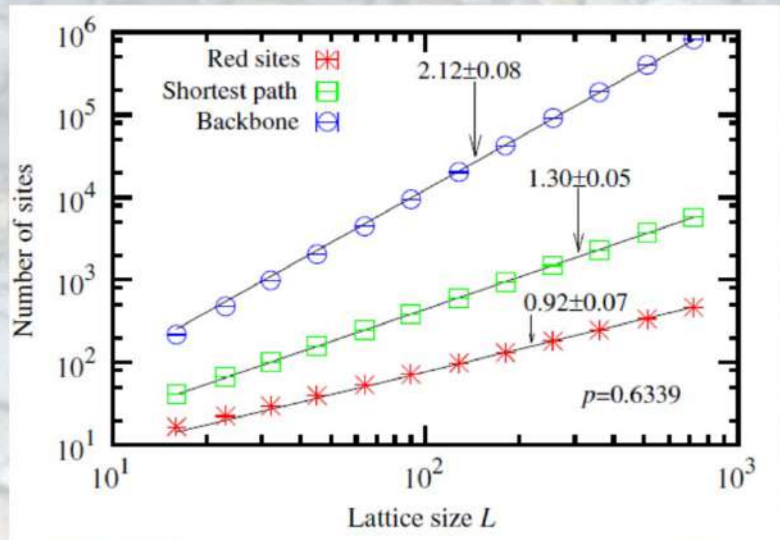


$$p_c = 0.6339 \pm 0.0004$$

$$\beta = 0.52 \pm 0.04$$

$$1/\nu = 0.92 \pm 0.01$$

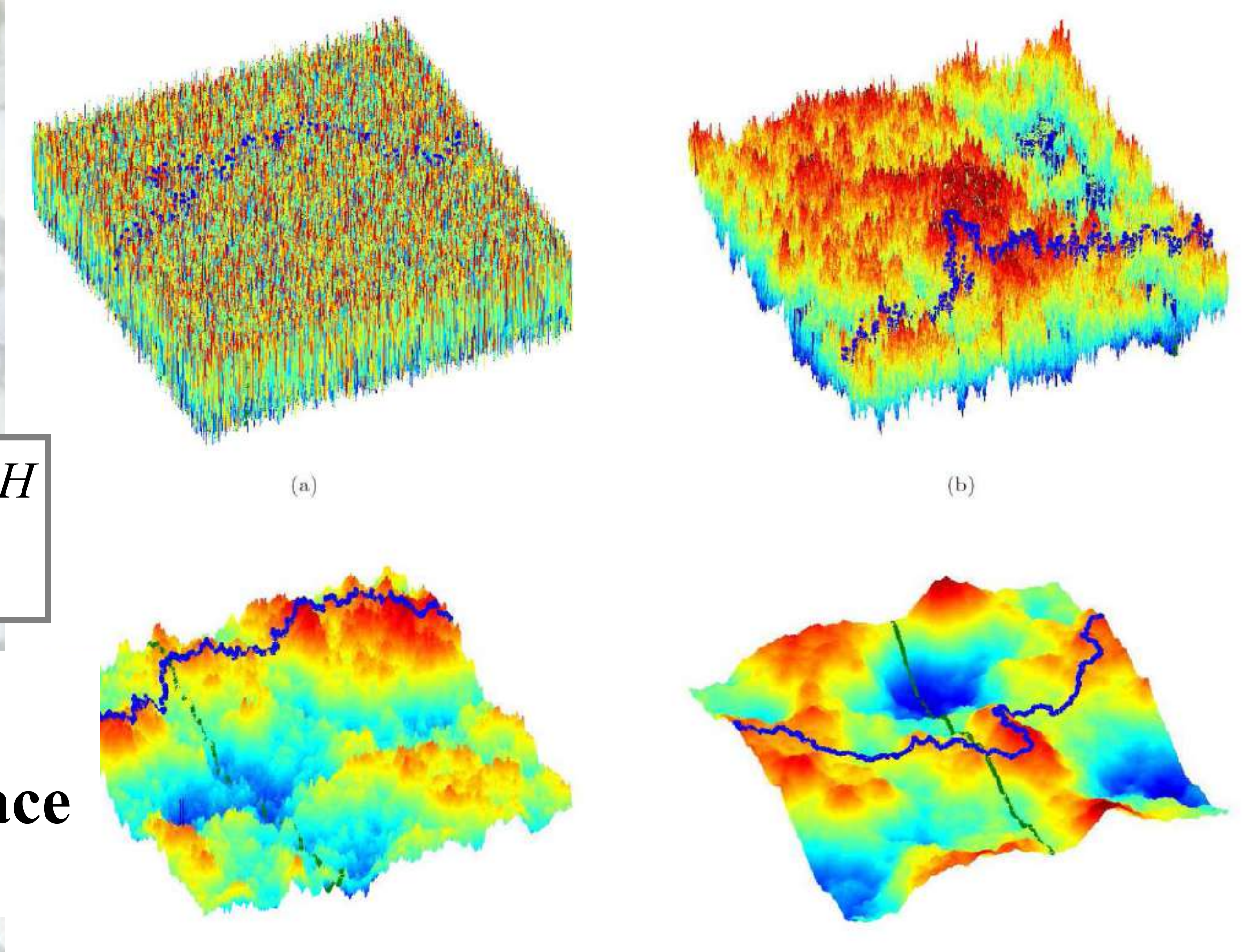
$$\gamma = 2.3 \pm 0.1$$



Correlated Landscapes

Artificial landscapes correlated through „fractional Brownian motion“

$$\left\langle \left(h(x)-h(y)\right)^2 \right\rangle \propto |x-y|^{2H}$$



H **is Hurst exponent**

H **= -1 uncorrelated surface**

H **= 0 Gaussian free field**

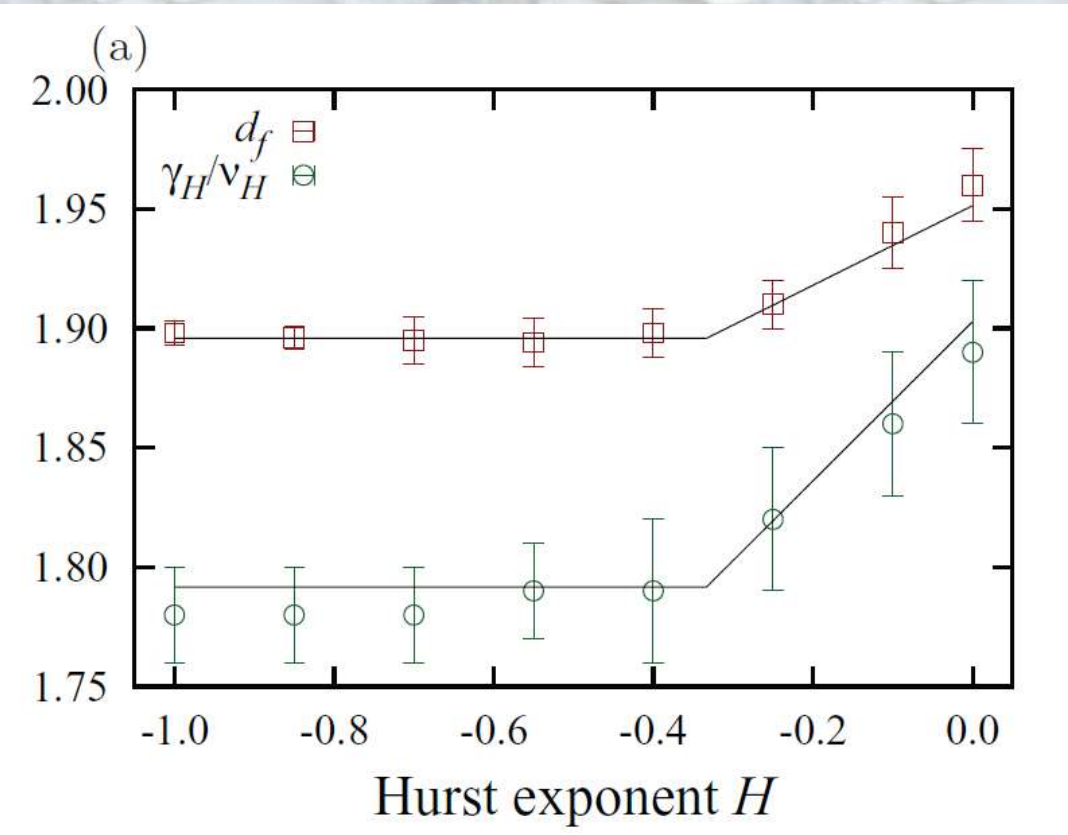
Fourier Filtering Method (Prakash et al, 1992)

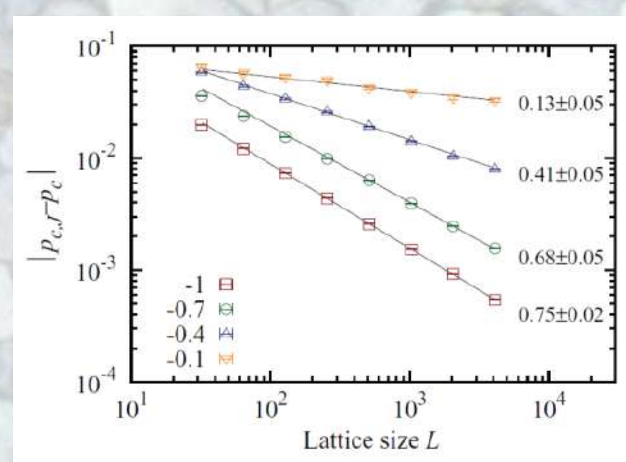
power spectrum

$$S(\omega) \propto |\omega|^{-2(H+1)}$$

Percolation on Correlated Landscapes

on triangular lattice p_c = ½ for all H



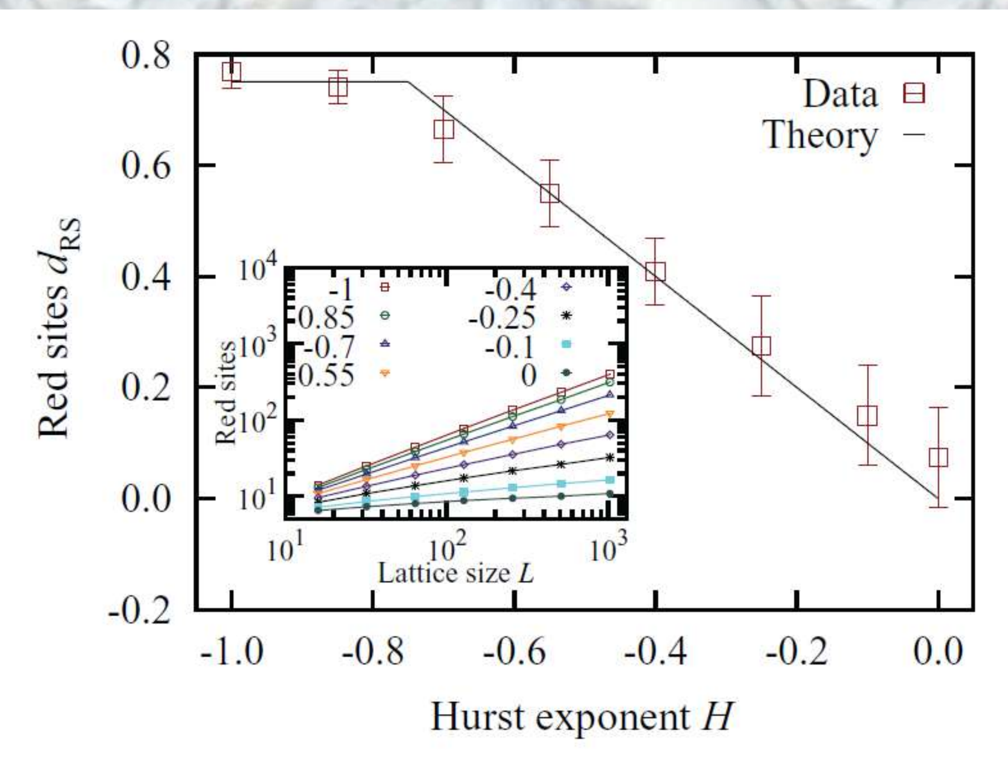


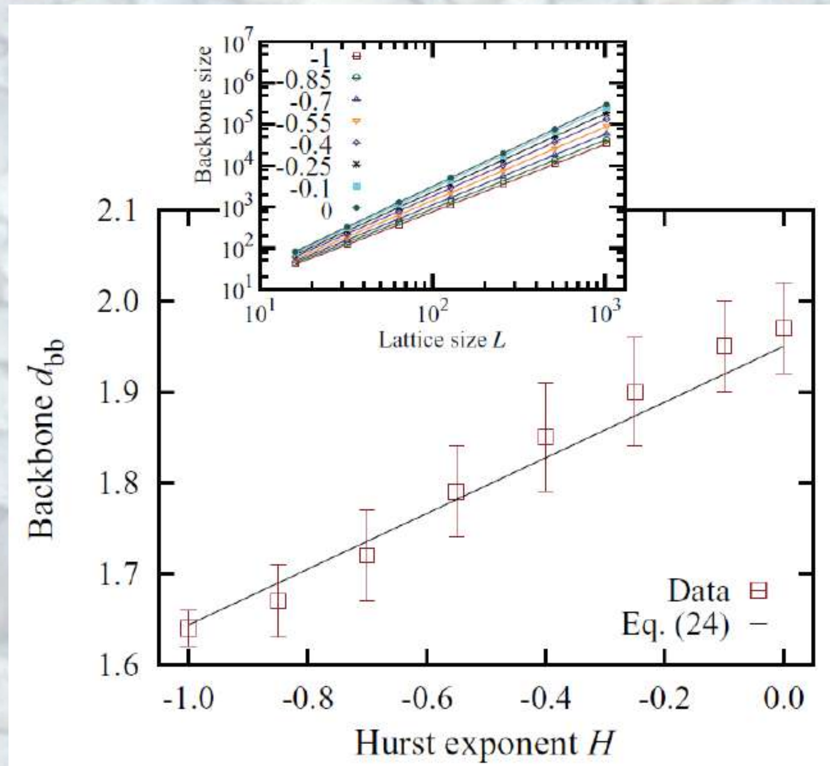
fractal dimension of the largest cluster and γ_H / ν_H as function of H

γ_H is exponent of second moment and ν_H of correlation length.

Percolation on Correlated Landscapes

at $p_c = ½$ fractal dimensions



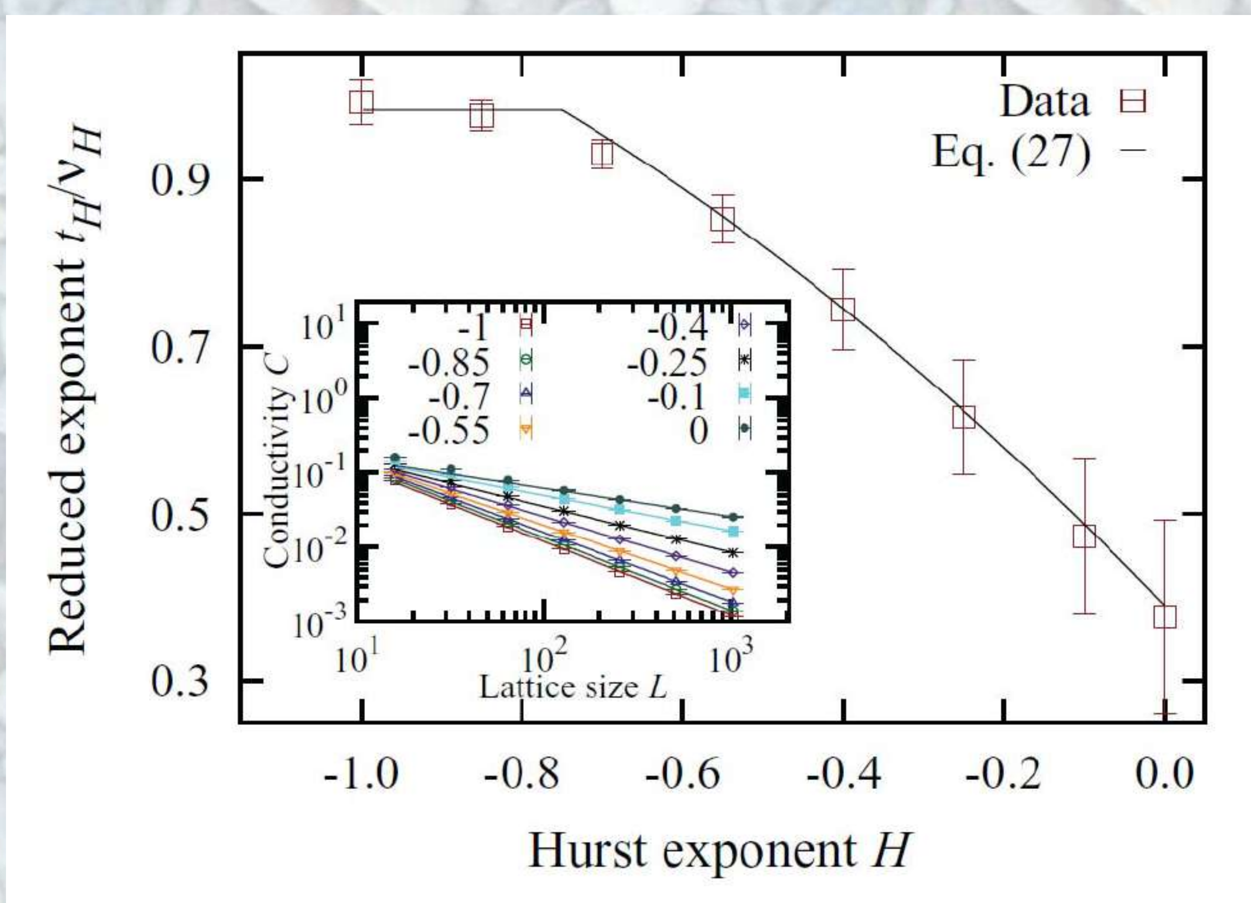


cutting bonds **backbone**

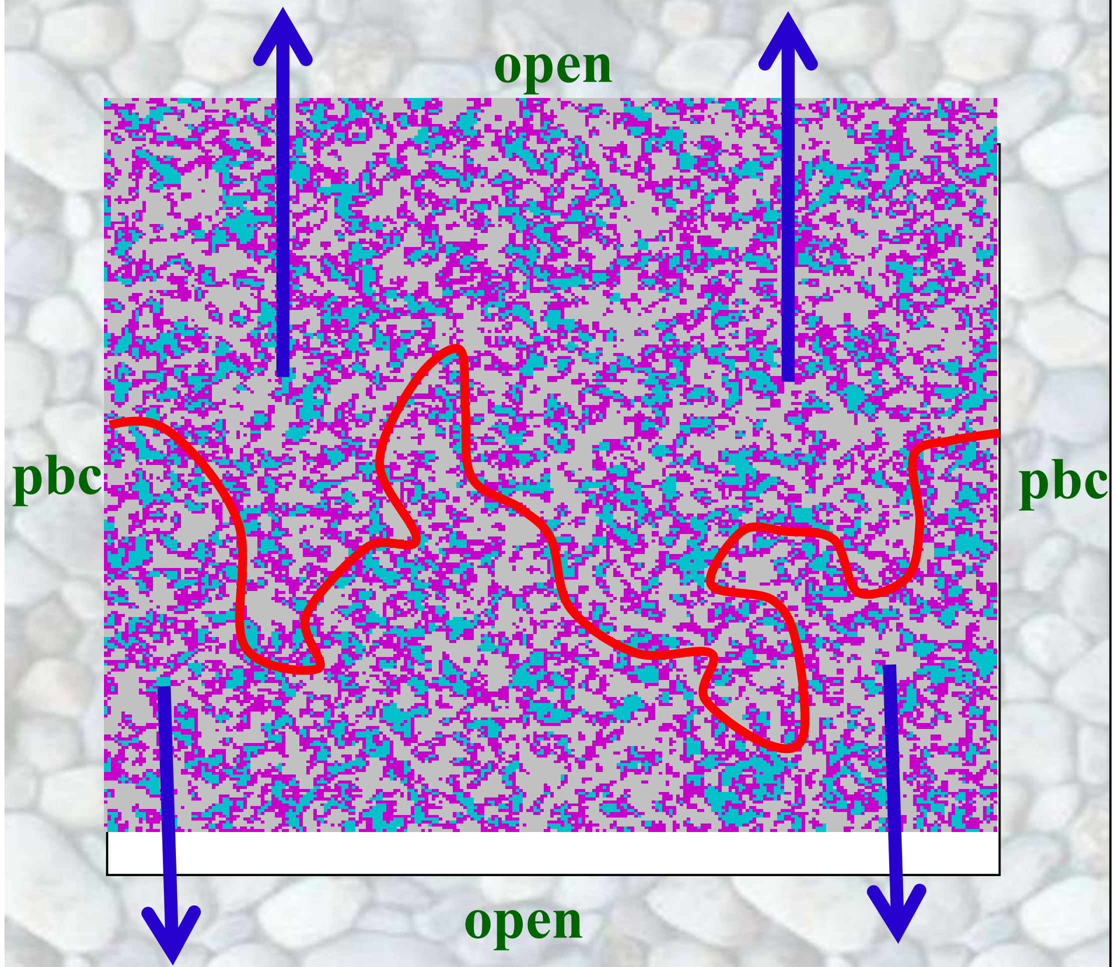
K.J. Schrenk, N. Posé, J.J. Kranz, L.V.M. van Kessenich, N.A.M. Araújo, H.J. Herrmann, Phys. Rev. E 88, 052102 (2013)

Percolation on Correlated Landscapes

exponent of electrical conductivity at $p_c = ½$



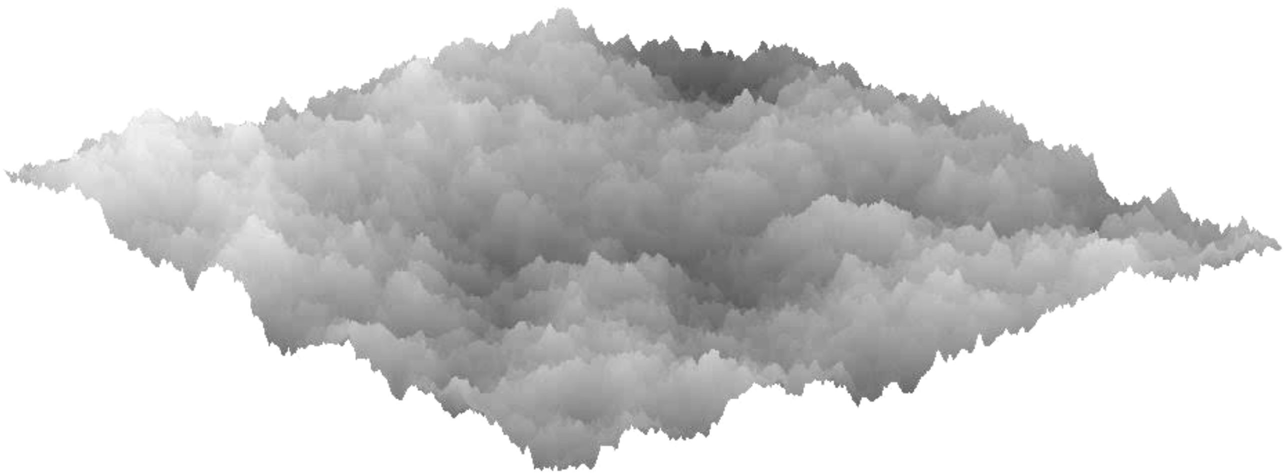
Watersheds



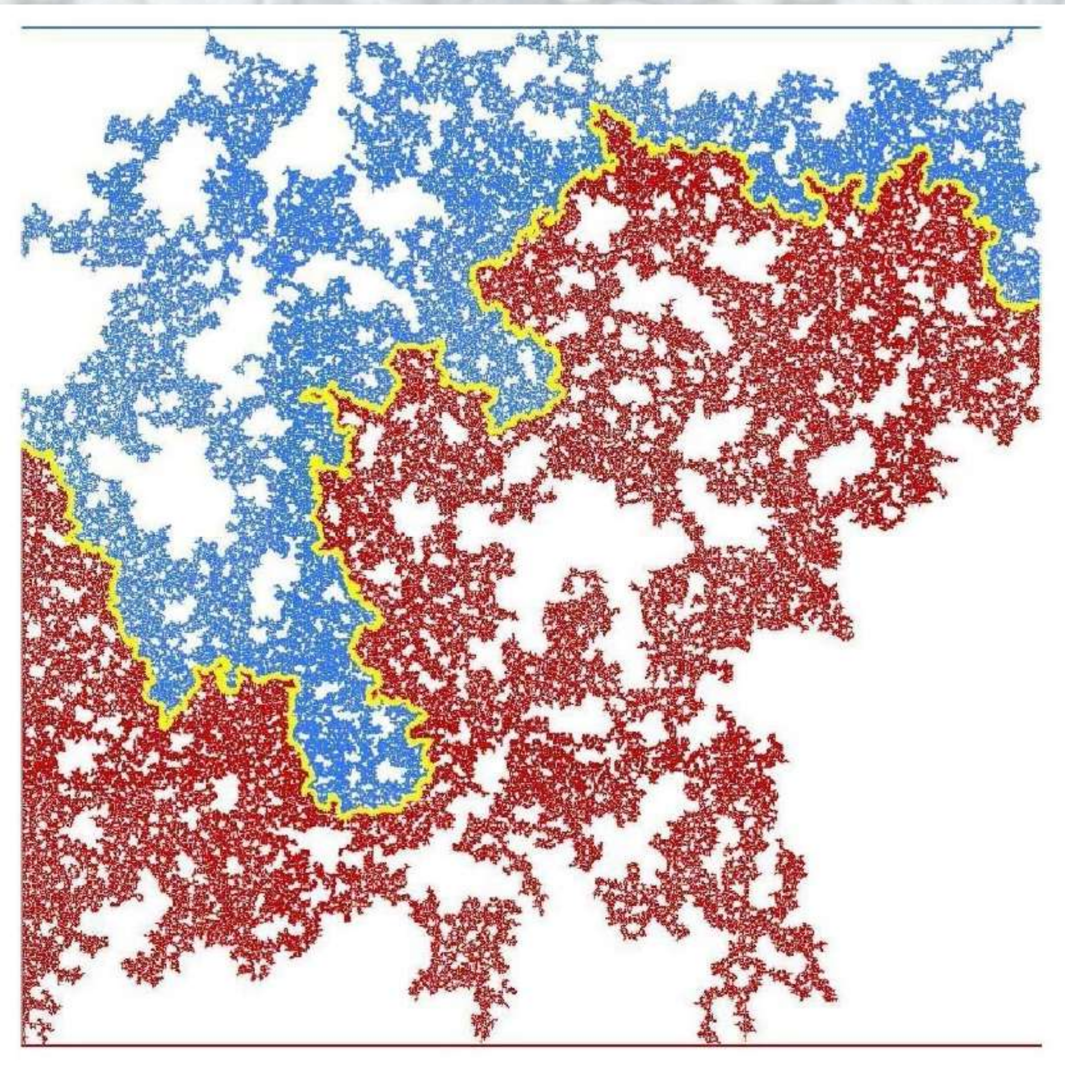
- Consider a landscape on a square lattice where h_i is the height at site *i*.
- Open b.c. on top and bottom and periodic b.c. between left and right.
- For each site *i* we determine if water from it would flow to the top or to the bottom.
- The watershed (or water divide) separates the sites for which it flows to the top from those for it flows to the bottom.

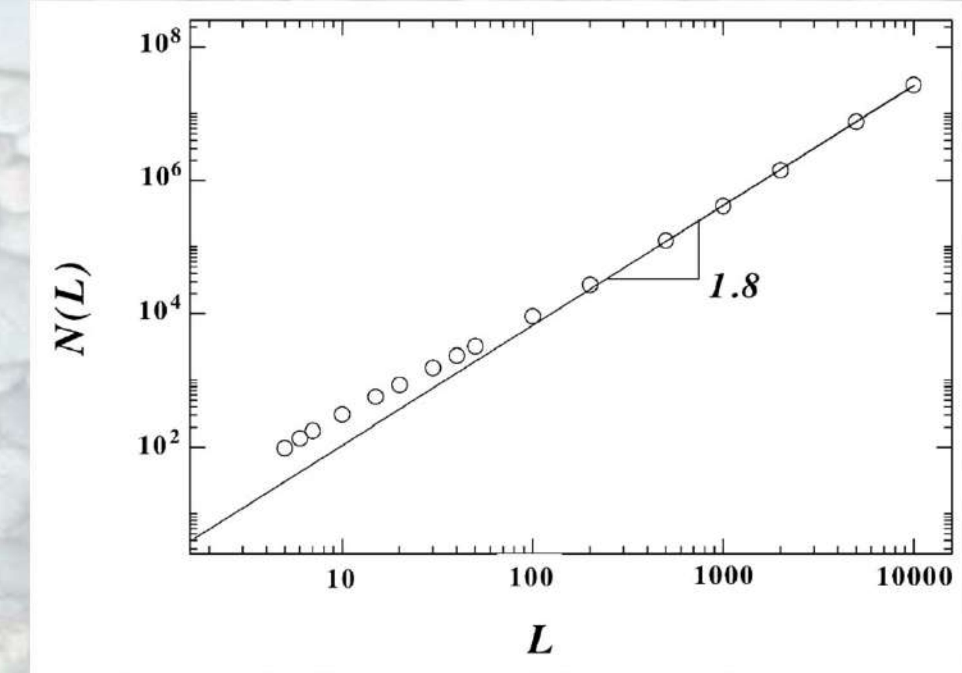
E. Fehr, J.S. Andrade, SD. da Cunha, L.R. da Silva, H.J. Herrmann, D. Kadau, C.F. Moukarzel, E.A. Oliveira, J. Stat. Mech. P09007 (2009)

Watersheds



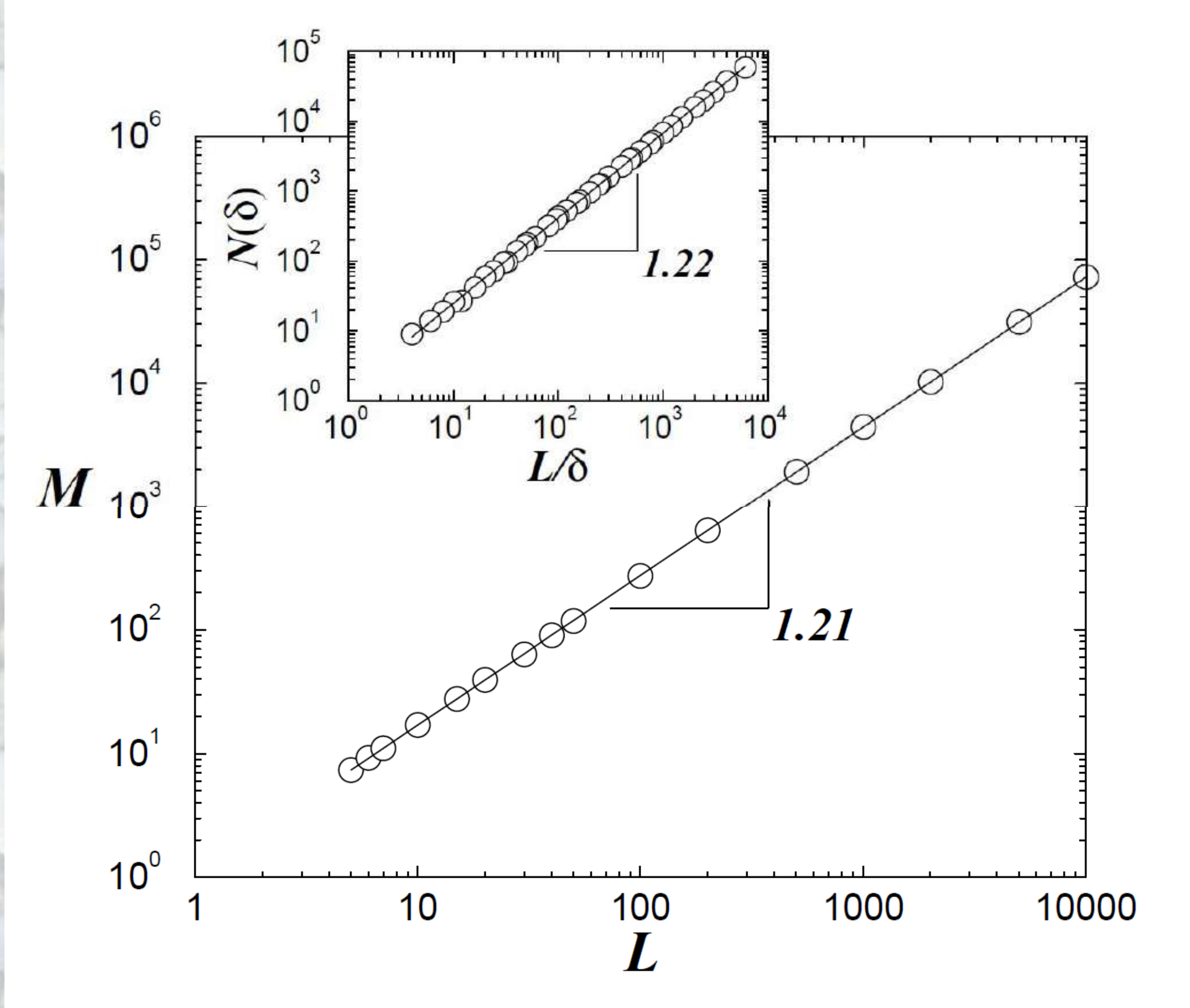
Numerical Calculation of Watersheds





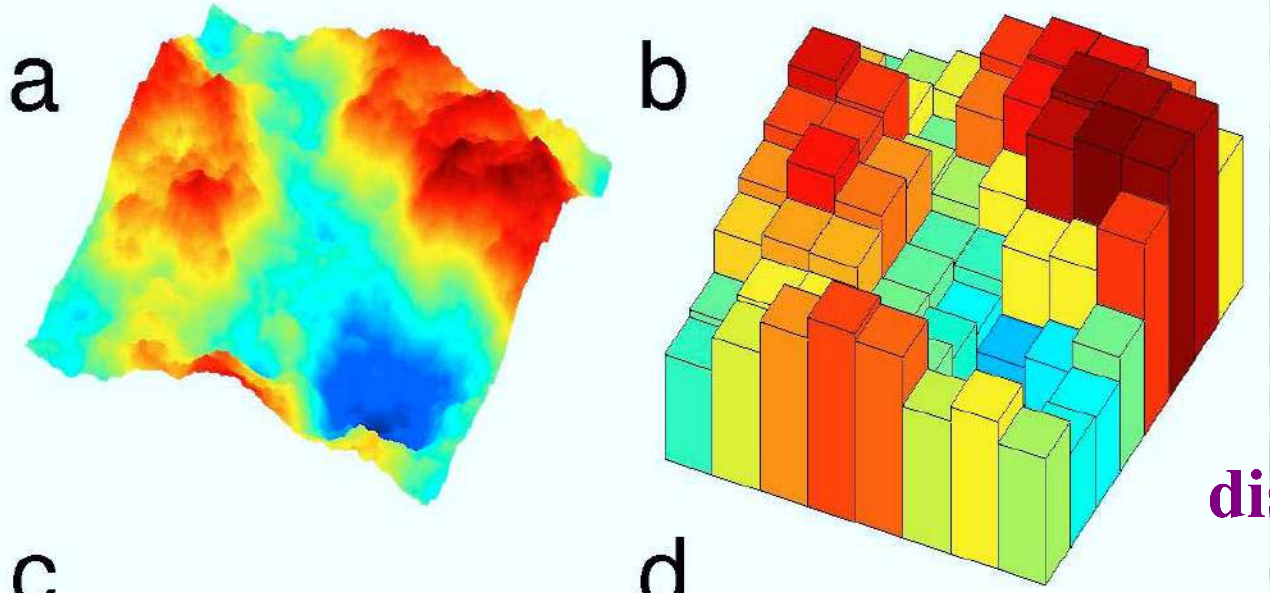
Watershed of random landscape

Local heights are randomly chosen from a homogeneous distribution.



Discrete landscapes

real landscape



discretization

DEM: discrete elevation map (course grained)

c

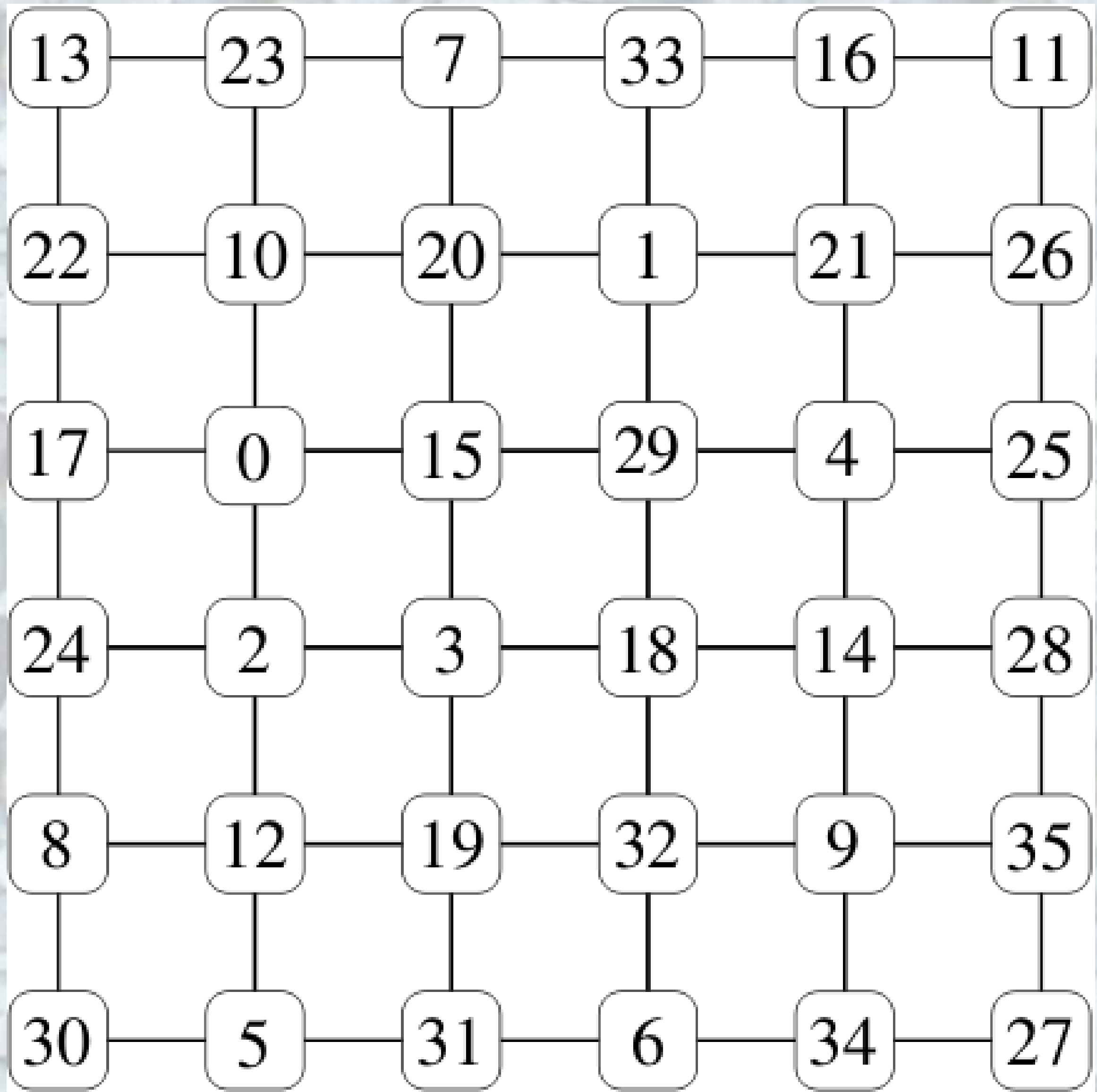
0.385	0.425	0.477	0.649	0.697	0.694	0.638	0.506
0.539	0.489	0.389	0.600	0.687	0.762	0.763	0.742
0.705	0.651	0.450	0.427	0.508	0.737	0.775	0.769
0.633	0.634	0.573	0.371	0.363	0.485	0.505	0.650
0.577	0.683	0.606	0.386	0.312	0.251	0.287	0.392
0.525	0.560	0.555	0.395	0.350	0.127	0.115	0.307
0.380	0.487	0.490	0.383	0.400	0.219	0.186	0.317
0.356	0.468	0.574	0.642	0.614	0.449	0.500	0.428

d

16	22	28	51	57	56	49	35
38	31	18	44	55	61	62	60
58	53	26	23	36	59	64	63
47	48	41	13	12	29	34	52
43	54	45	17	8	5	6	19
37	40	39	20	10	2	1	7
14	30	32	15	21	4	3	9
11	27	42	50	46	25	33	24

ranked surface

Ranked surface



Size of Phase Space

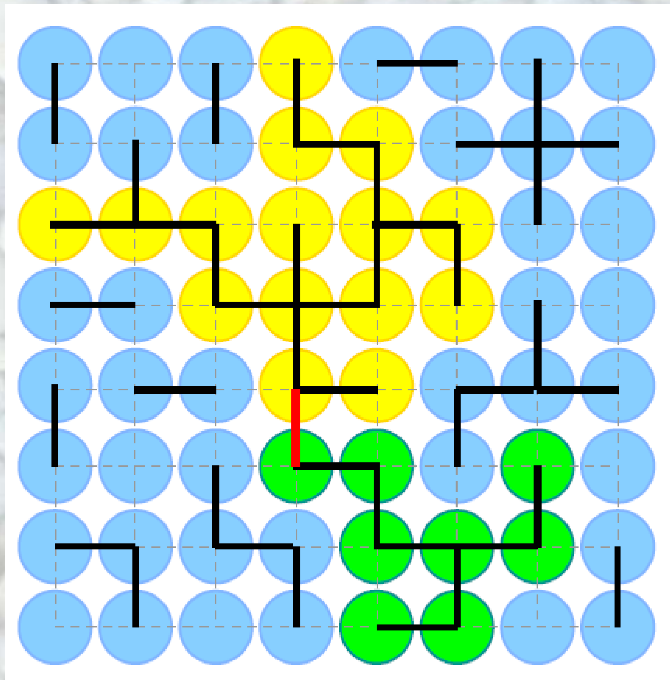
N is the number of sites

Number of configurations of usual percolation 2^N

Number of configurations of ranked percolation $N!$

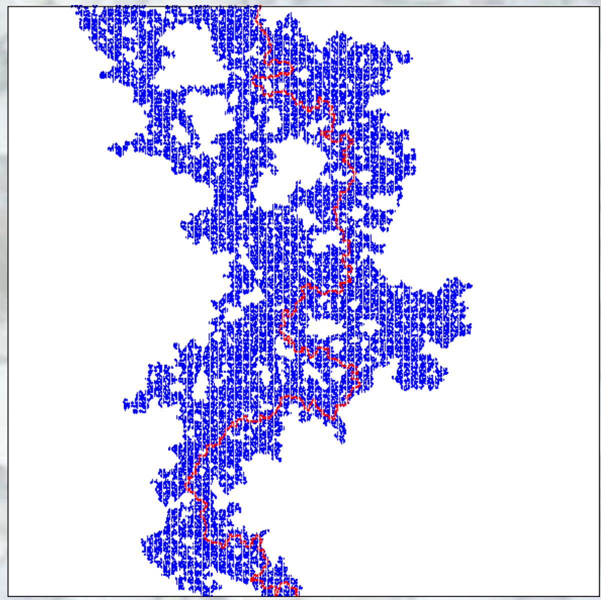
Same universality class

bridge percolation

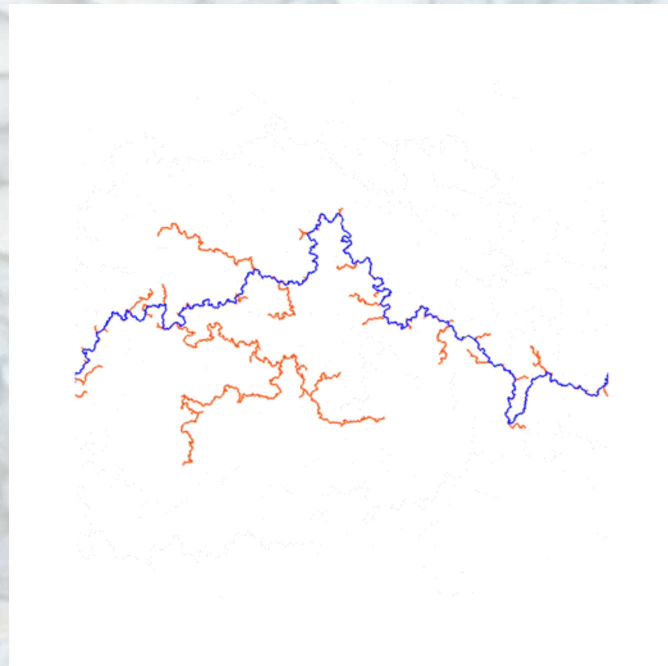


K.J. Schrenk, N.A.M. Araújo,
J.S. Andrade Jr., H.J.H.,
Sci. Rep. 2, 348 (2012)

shortest path on loop-less percolation



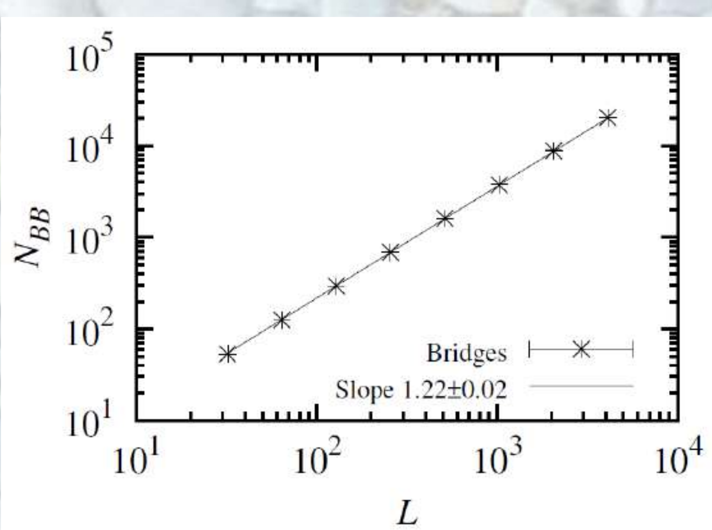
optimal path crack



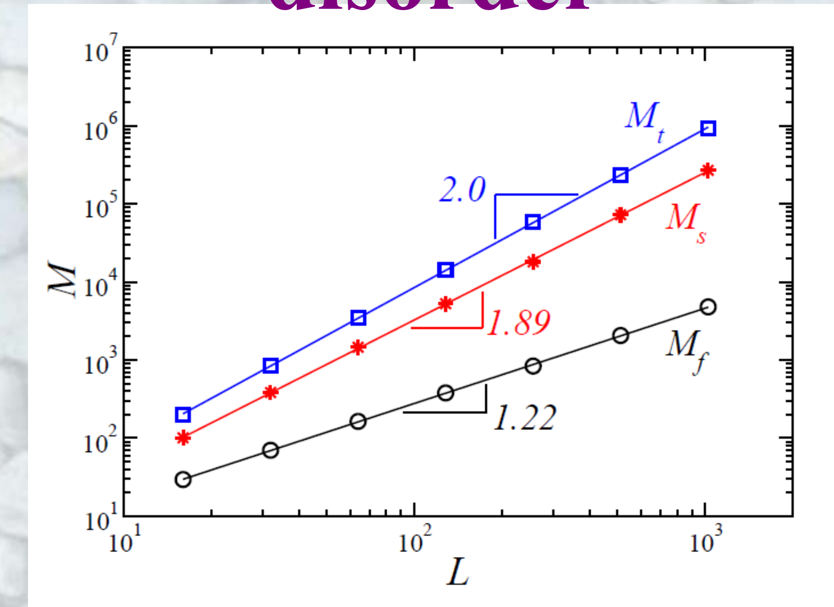
J.S. Andrade Jr., E. Oliveira,
A. Moreira and HJH,
Phys.Rev.Lett. 103, 225503
(2009)

Same universality class

Two invading liquids touching

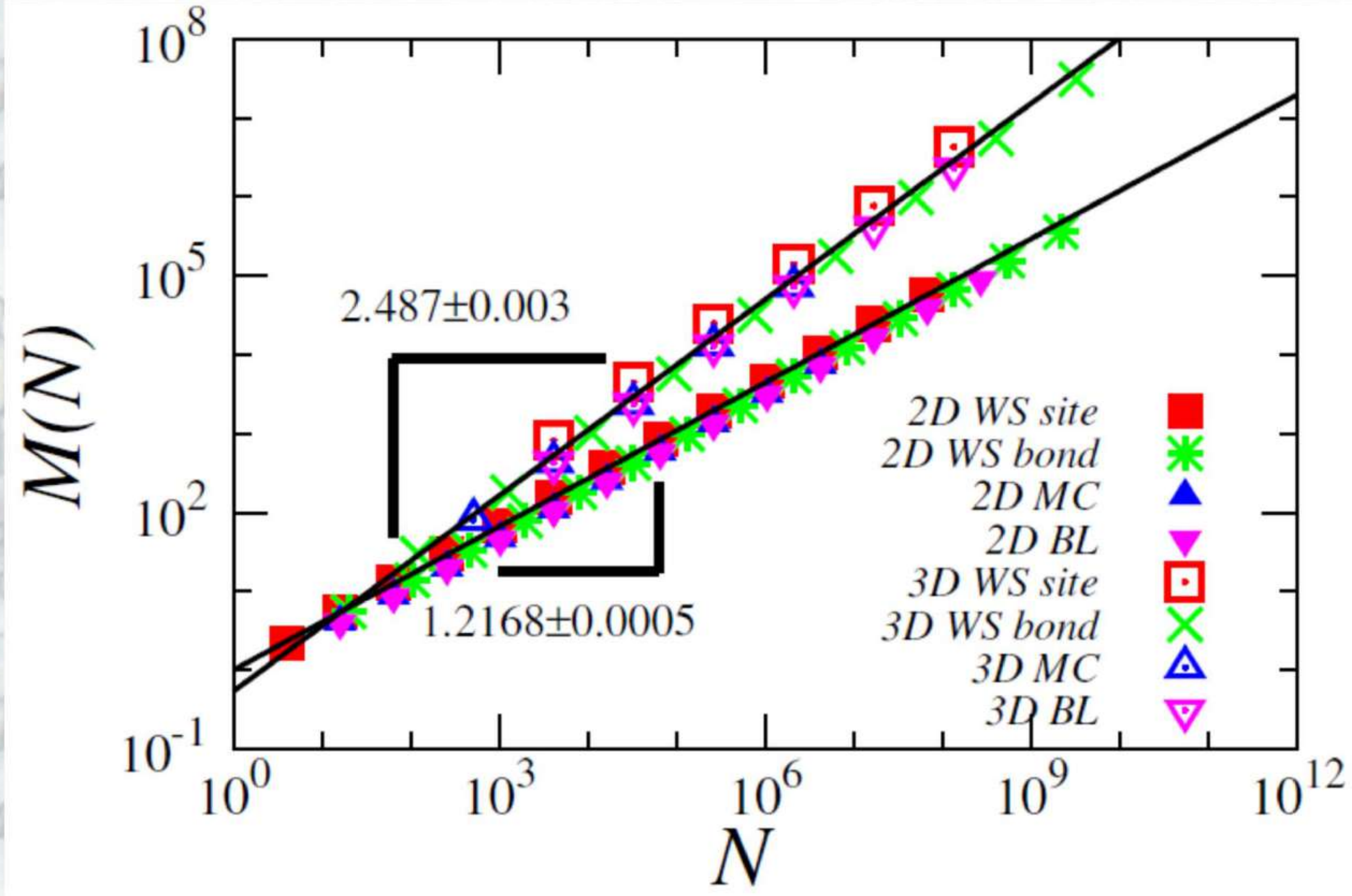


Fuses in infinite disorder



A.A. Moreira, C.L.N. Oliveira, A. Hansen, N.A.M. Araújo, H.J.H., J.S. Andrade Jr, Phys. Rev. Lett. 109, 255701 (2012)

High precision calculation

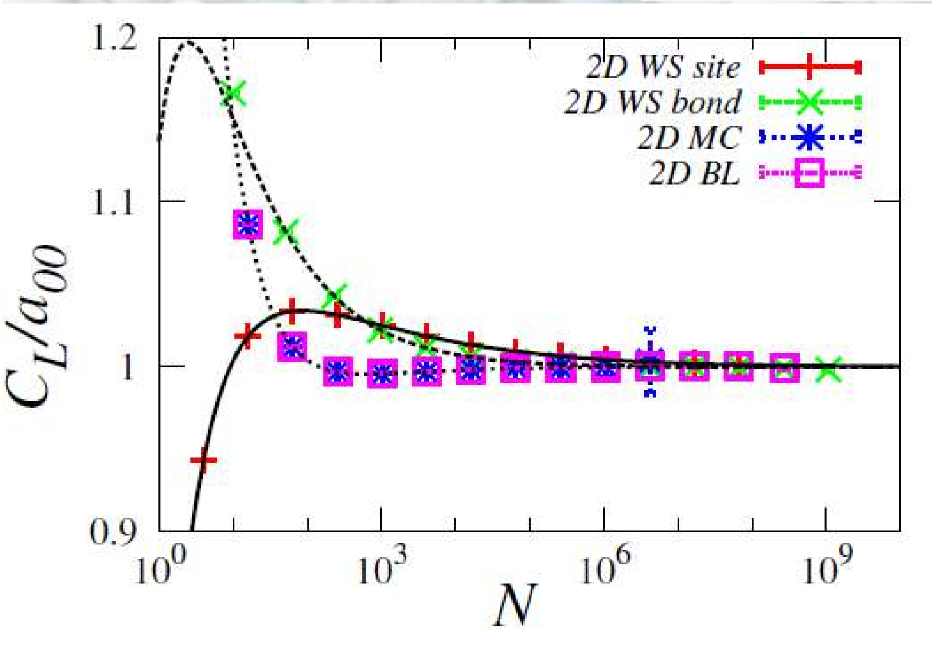


E. Fehr, K.J. Schrenk, N.A.M. Araújo, D. Kadau, P. Grassberger, J.S. Andrade Jr., H.J.H. Phys. Rev.E 86, 011117(2012)

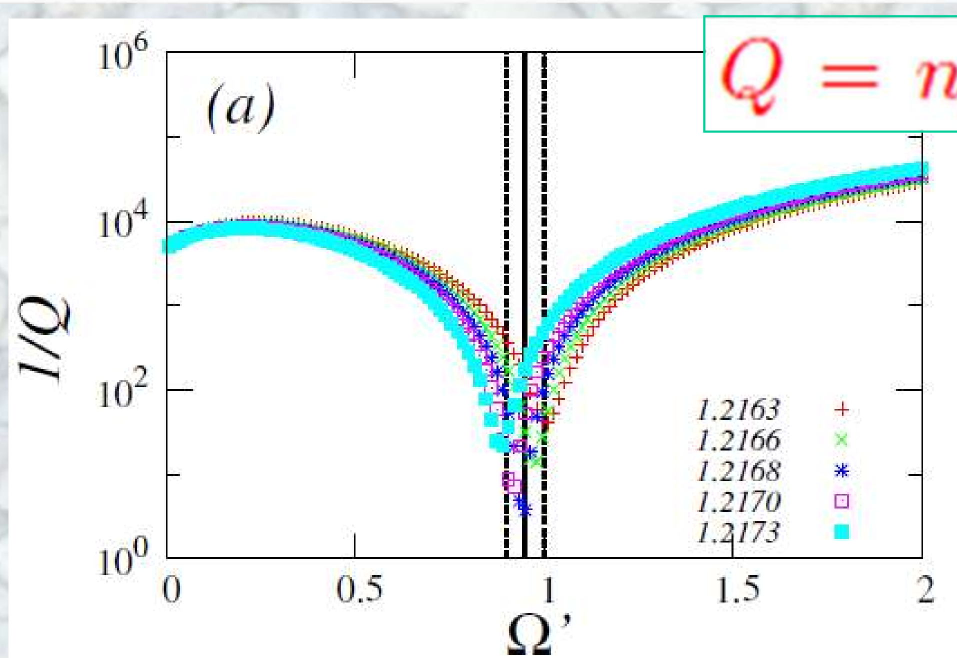
Corrections to scaling

$$M_L = L^{d_f} C_L$$

$$C_L^{2D} = a_{00} + a_{11} L^{-\omega} + a_{21} L^{-\Omega} + a_{22} L^{-\Omega-1}$$



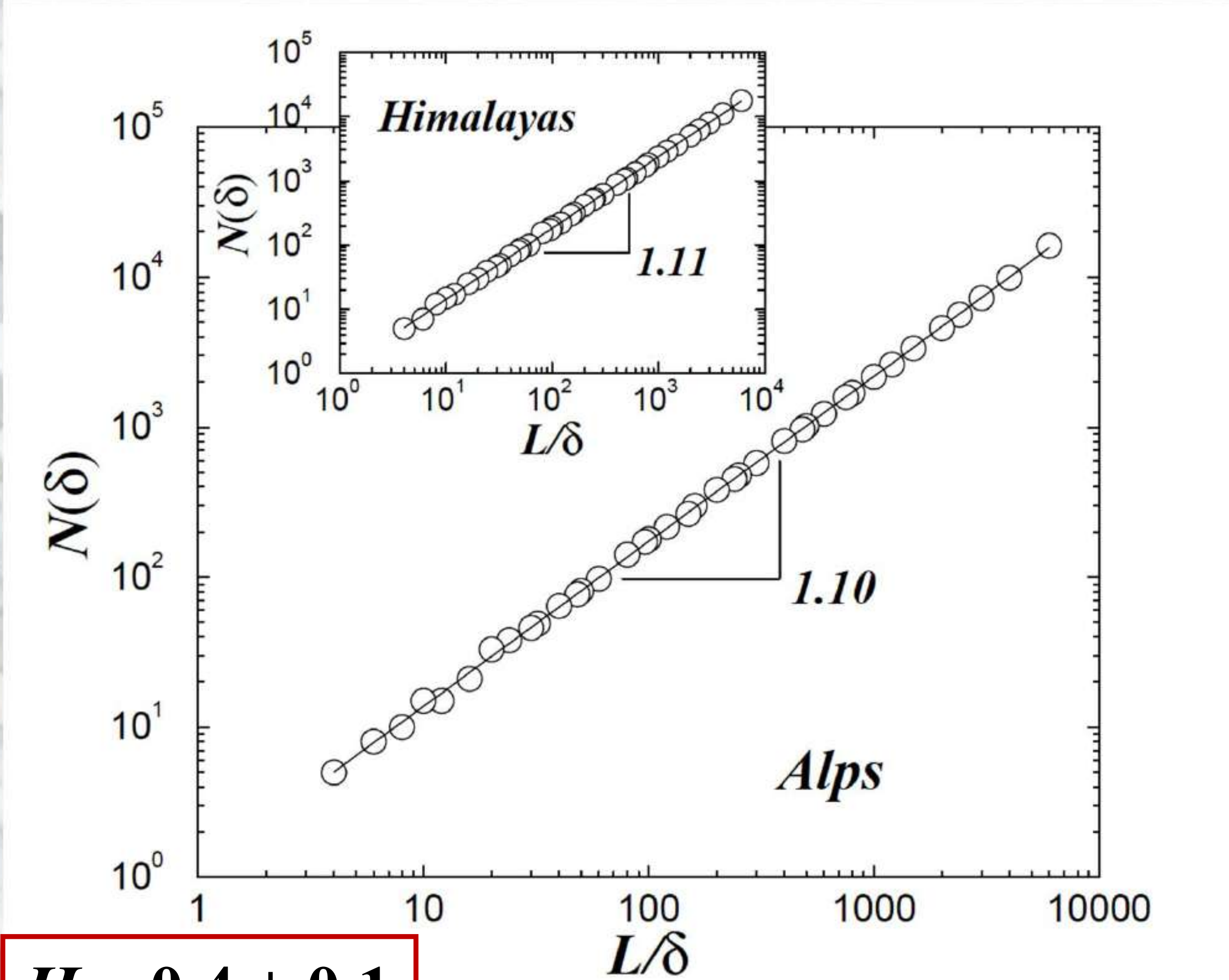
$$Q = n/\chi^2$$



model	d	d_f	Ω
WS bond	2	1.2168±0.0005	0.95±0.05
WS site	2	1.21705±0.00075	0.91±0.19
BL	2	1.21655±0.0015	0.87±0.08
MC	2	1.21655±0.0045	0.86±0.11

E. Fehr, K.J. Schrenk, N.A.M. Araújo, D. Kadau, P. Grassberger, J.S. Andrade Jr., H.J.H. Phys. Rev.E 86, 011117(2012)

Watersheds on natural landscapes



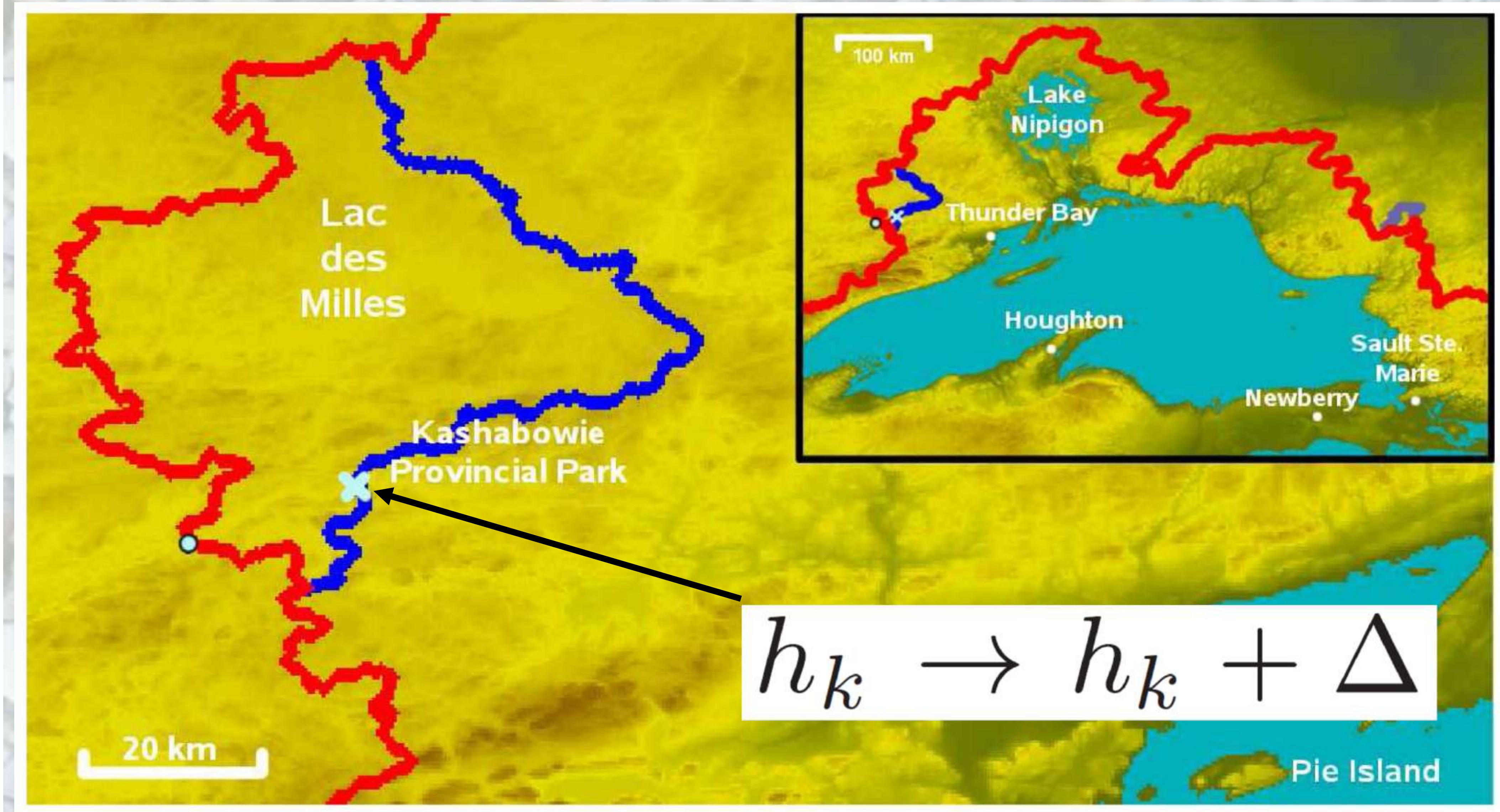
$H = 0.4 \pm 0.1$

Landscapes have a spatial power-law correlation described by a Hurst exponent *H*:

$$\left\langle \left(h(x)-h(y)\right)^2 \right\rangle \propto |x-y|^{2H}$$

E. Fehr, J. S. Andrade Jr., S. D. da Cunha, L. R. da Silva, H.J.H., D. Kadau, C. F. Moukarzel and E. A. Oliveira, J. Stat. Mech. , P09007 (2009)

Perturbations on Watersheds



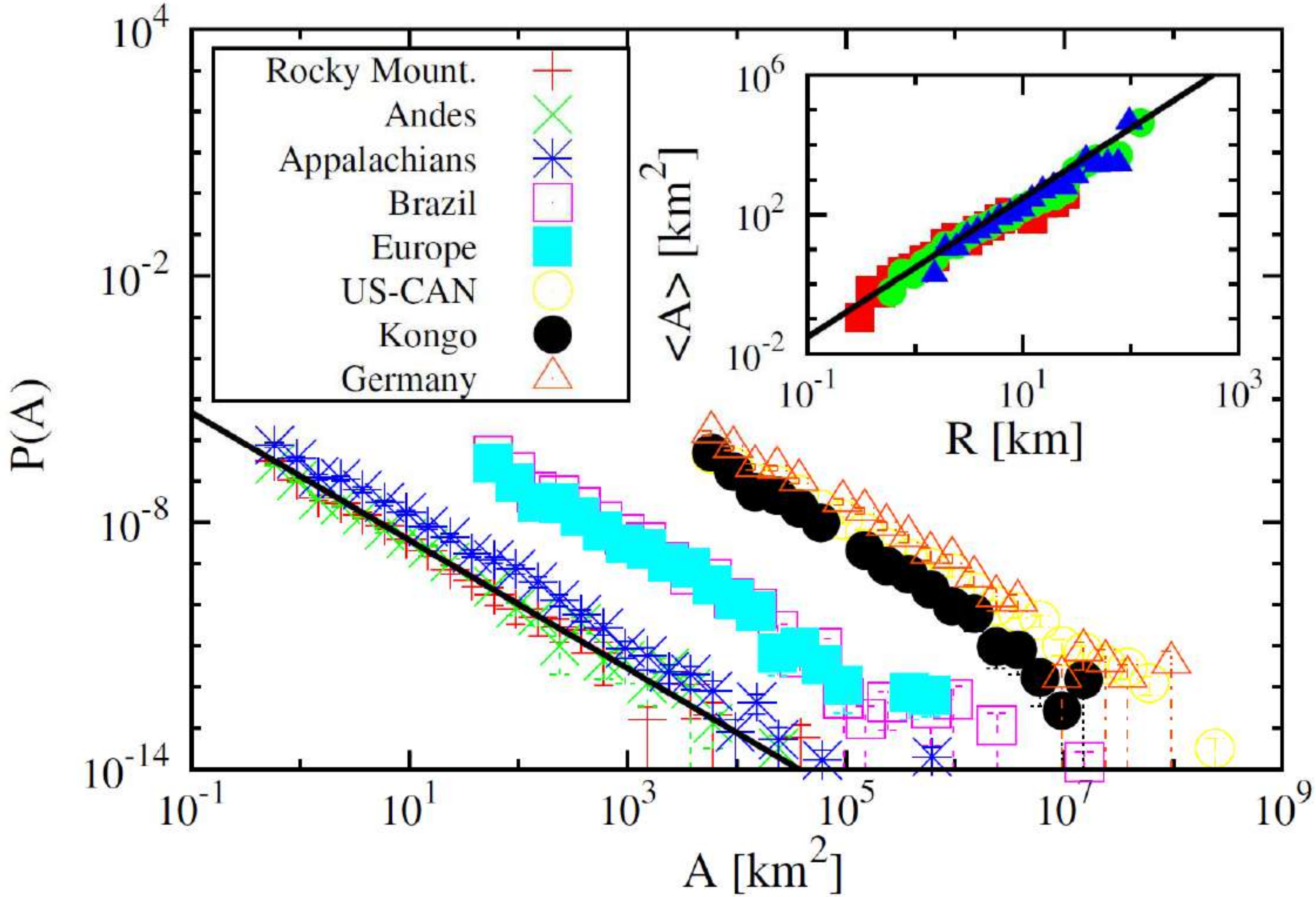
E. Fehr, D. Kadau, J.S. Andrade Jr., HJH, Phys. Rev. Lett 106, 048501(2011)

Perturbations on Watersheds

Distribution of areas A for different landscapes following:

$$P(A) \propto A^{-\beta}$$

$\beta = 1.65$



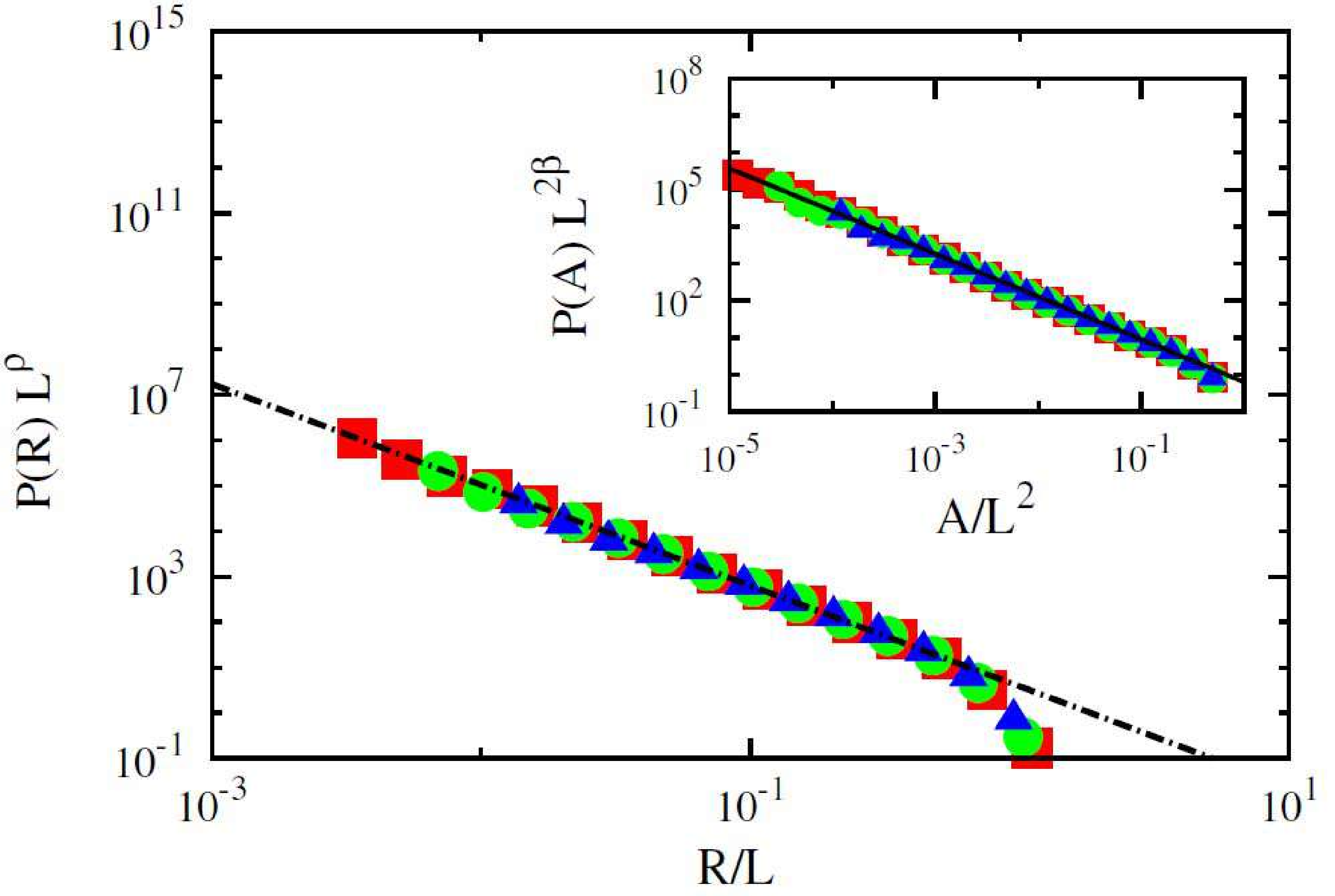
R = distance between outlets ;   A = area

Perturbations on Watersheds

Scaling of the distribution of R and of A with system size for an artificial landscape with uniformly distributed heights.

$\rho = 2.21$

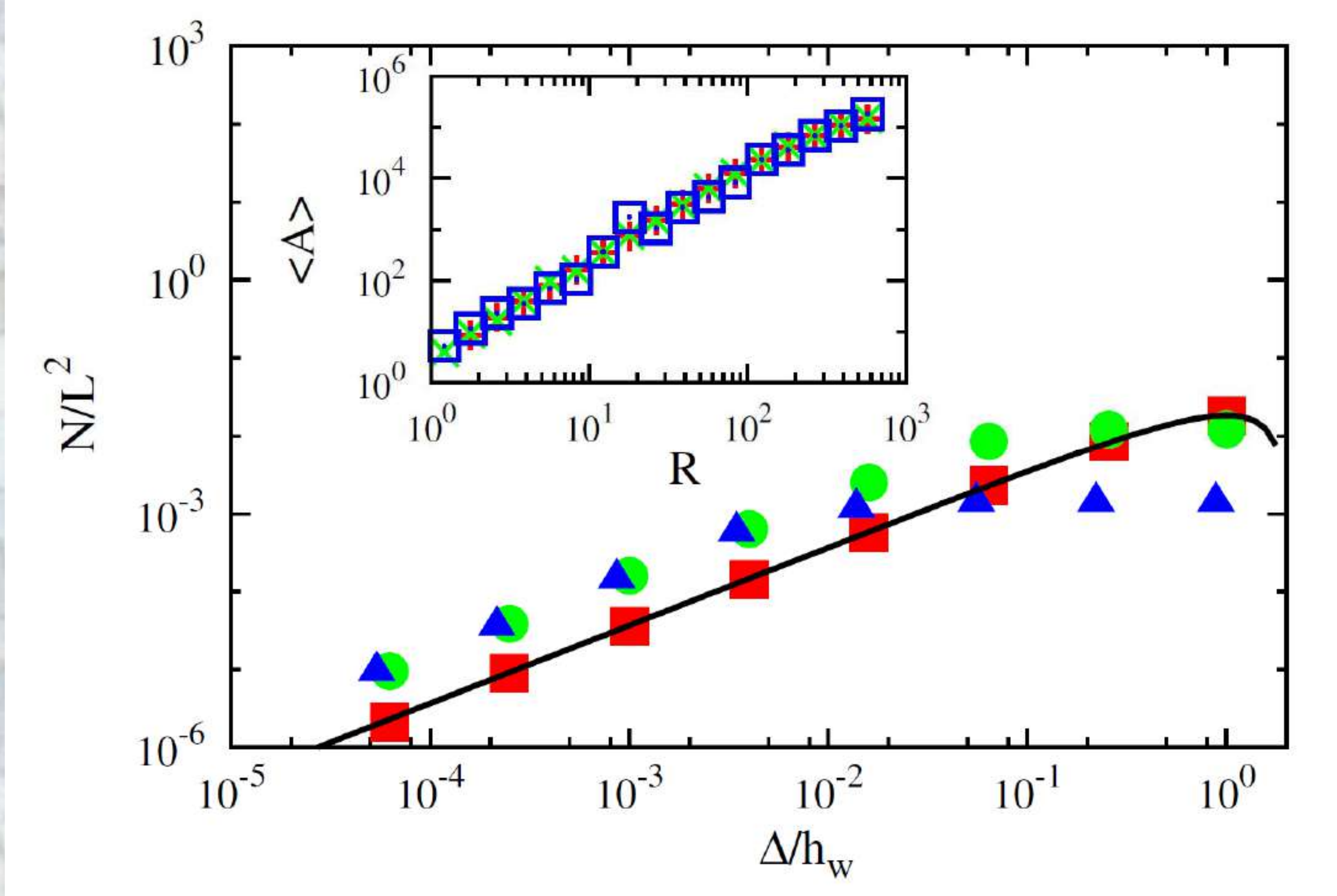
$\beta = 1.16+/-0.03$



R = distance between outlets ; A = area

Perturbations on Watersheds

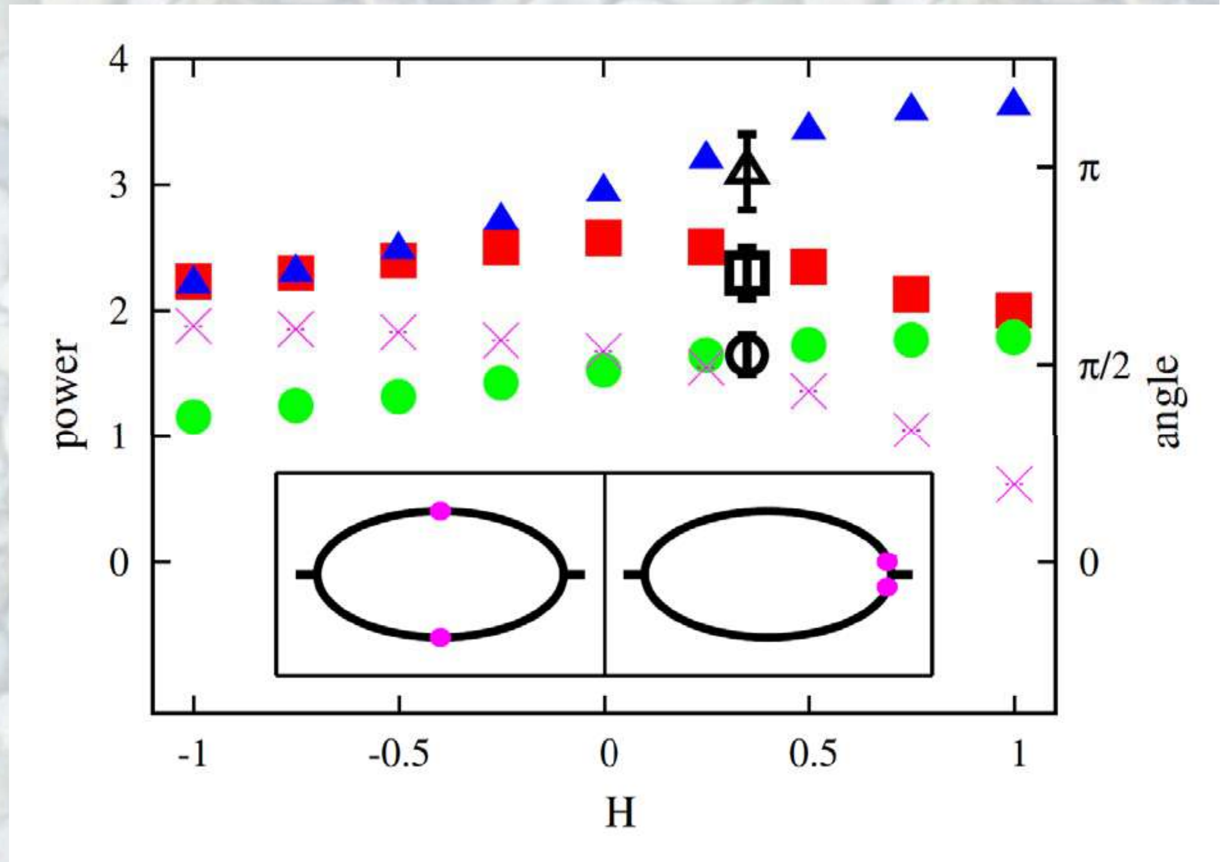
Number of sites *N* on which a perturbation makes a change of the watershed as function of the strength Δ of the perturbation.



E. Fehr, D. Kadau, J.S. Andrade Jr., HJH, Phys. Rev. Lett 106, 048501(2011)

Perturbations on Watersheds

Dependence of the exponents α (squares) β (circles) and ρ (triangles) on the Hurst exponent for artificial correlated landscapes.



$$P(R) \sim R^{-\rho} \qquad P(A \mid R) \propto A^{-\alpha}$$

Schramm-Loewner Evolution (SLE)

Special mapping of a loopless path in complex space to a scalar random time series, called «driving function».

If fractal path conformally invariant and Markovian, then the driving function is a Brownian walk and its diffusivity κ is related to the fractal dimension d_f of the path through:

2d

$$d_f = 1 + \frac{\kappa}{8}$$

$\kappa = 2$	loop erased random walk
$\kappa = 8/3$	self-avoiding walk
$\kappa = 3$	hull of critical Ising clusters
$\kappa = 4$	Gaussian free field
$\kappa = 6$	perimeter of critical percolation clusters

Schramm-Loewner Evolution (SLE)

conformally invariant and Markov property

$g_t(z) : \mathbb{H} \to \mathbb{H}$ is a conformal mapping following the Loewner equation:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}, \qquad g_0(z) = z$$

$$\xi_t = \sqrt{\kappa} B_t$$ **is the „driving function“**

where B_t is a 1d Brownian motion

Generation of driving function

«zipper algorithm with vertical slit discretization»:

$$f_k(z) = g_k^{-1}(z)$$

given discrete (complex) values of the path: γ_k

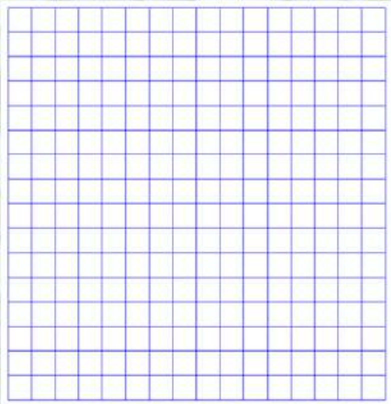
$$f_k(z) = i\sqrt{-\mathrm{Im}\{\omega_k\}^2 - (z - \mathrm{Re}\{\omega_k\})^2}.$$

$$\omega_k = f_{k-1} \circ f_{k-2} \circ \ldots \circ f_1(\gamma_k) \qquad \omega_1 = \gamma_1,$$

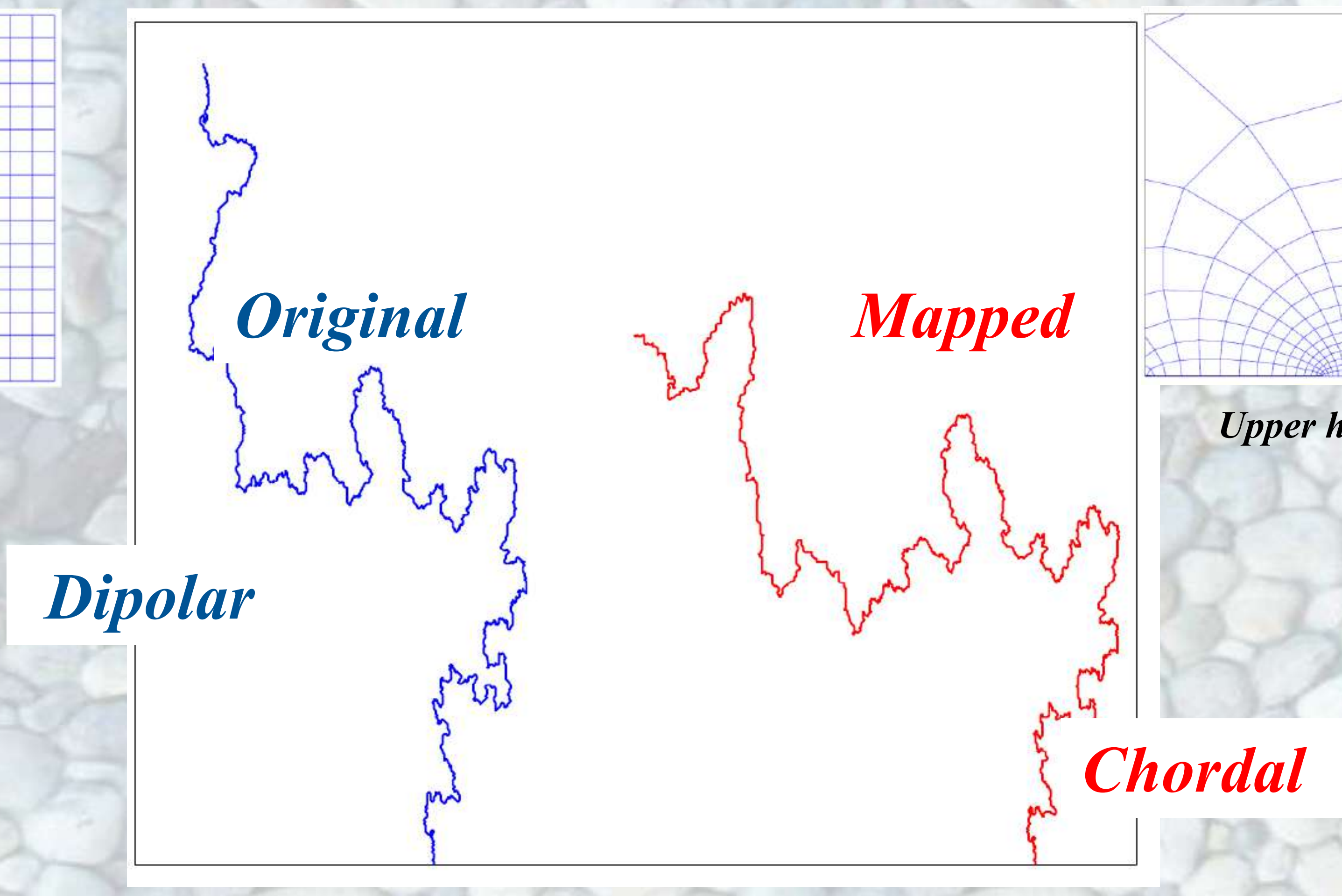
$$t_k = \frac{1}{4}\sum_{j=1}^{k} \mathrm{Im}\{\omega_j\}^2 \qquad U_{t_k} = \sum_{j=1}^{k} \mathrm{Re}\{\omega_j\},$$

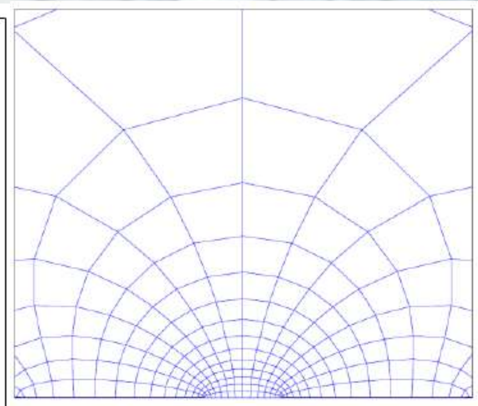
T. Kennedy, J.Stat.Phys. 131, 803 (2008)

Schwarz-Christoffel mapping



Square Lattice

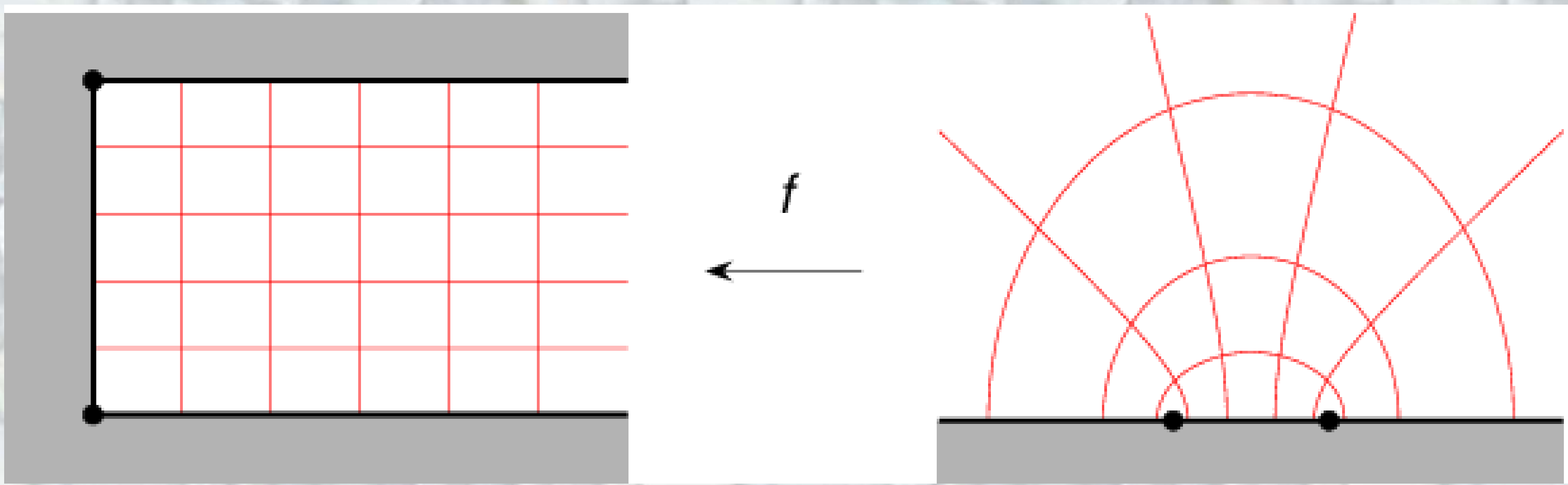




Upper half-plane

T. A. Driscoll, *ACM Trans. Math. Softw.* 22, 168 (1996)

Schwarz-Christoffel mapping



Let $\mathbb{C}^+$ denote the open complex upper half-plane, and define f on $\mathbb{C}^+$

$$f(z) = a + c \int_0^z \prod_{k=1}^{n-1} (s - x_k)^{\beta_k} \, ds,$$

for some real $x_1, \ldots, x_{n-1}$ satisfying

$$x_1 < x_2 < \cdots < x_{n-1} < x_n = \infty$$

Driving function for Watershed

Direct SLE

$$D_i(z) = \xi(t) + \sqrt{(z - \xi(t))^2 + 4\delta t}$$

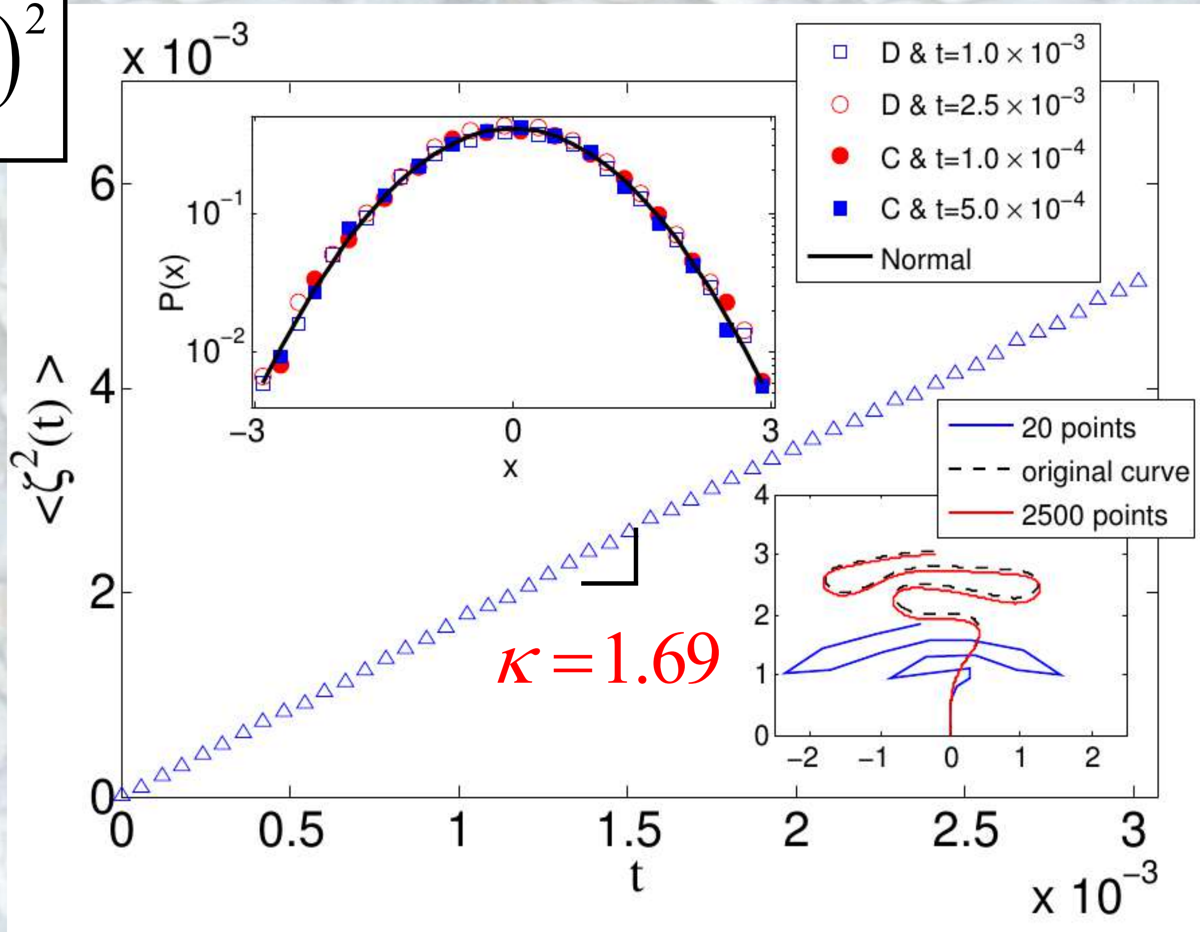
$$t_i = t_{i-1} + \delta t \ , \quad \delta t = \frac{1}{4}(\mathrm{Im}\, z_i)^2$$

$$\xi(t_i) = \mathrm{Re}\, z_i$$

$$x = \frac{\xi(t)}{\sqrt{\kappa t}}$$

$$\kappa = 1.69 \pm 0.05$$

$$d_f = 1 + \frac{\kappa}{8} \approx 1.21$$

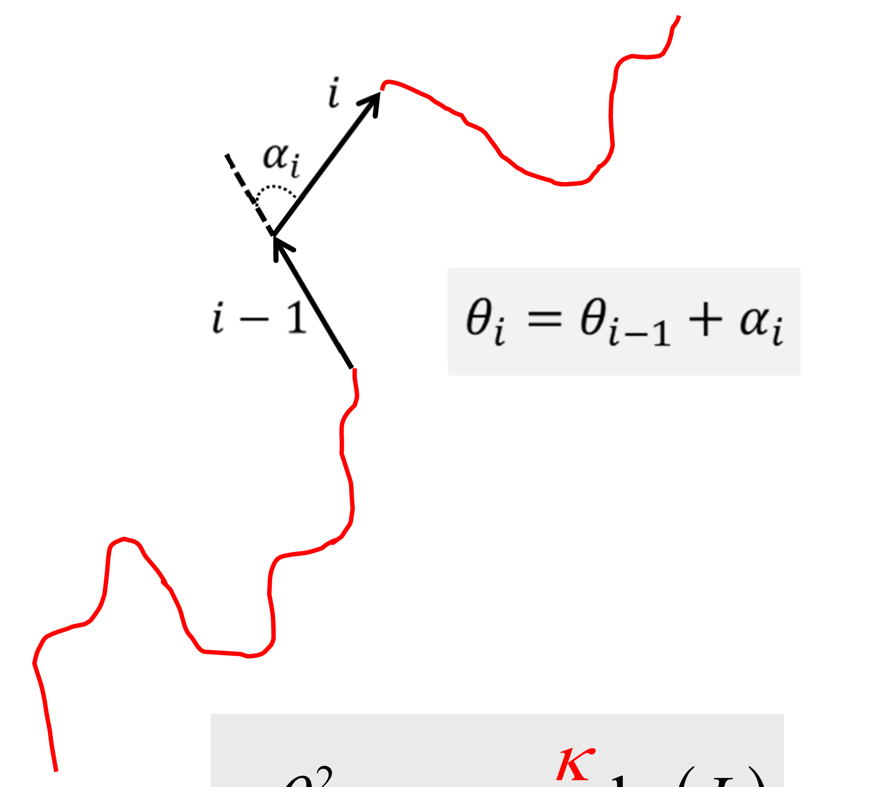


E. Daryaei, N. A. M. Araujo, K. J. Schrenk, S. Rouhani, and H. J. H., Phys. Rev. Lett. 109, 218701 (2012)

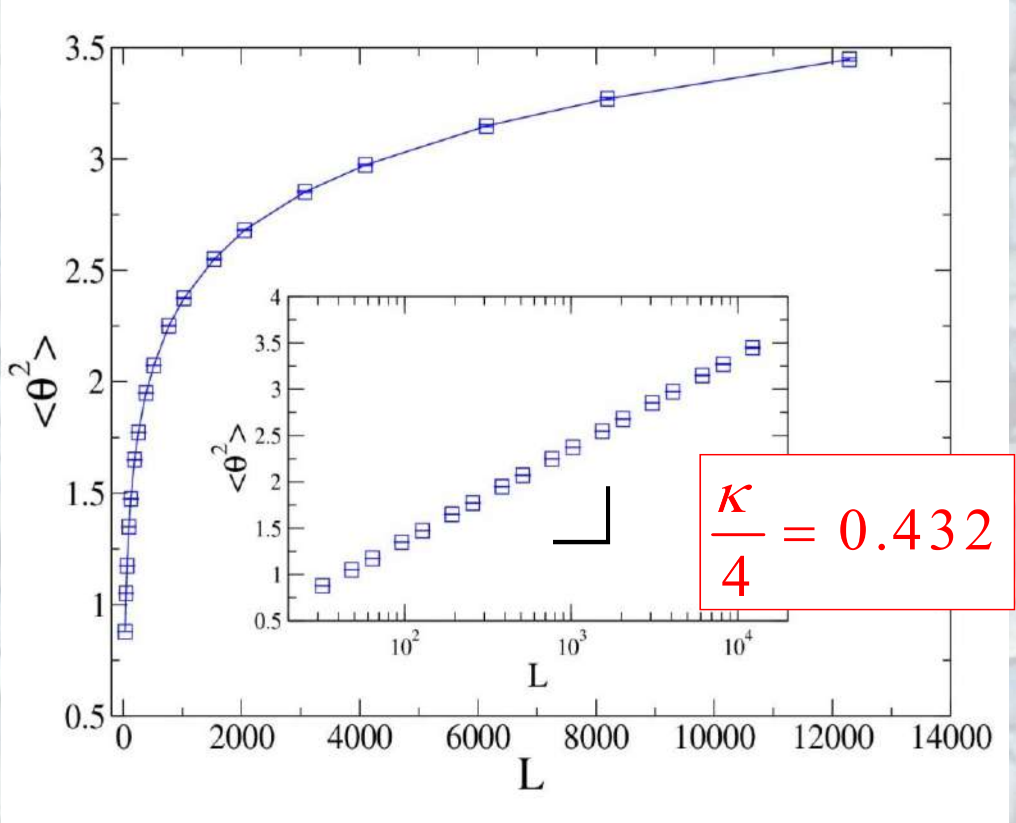
Winding angle for Watershed

$d_f \approx 1.216$

$\kappa = 1.728 \pm 0.008$

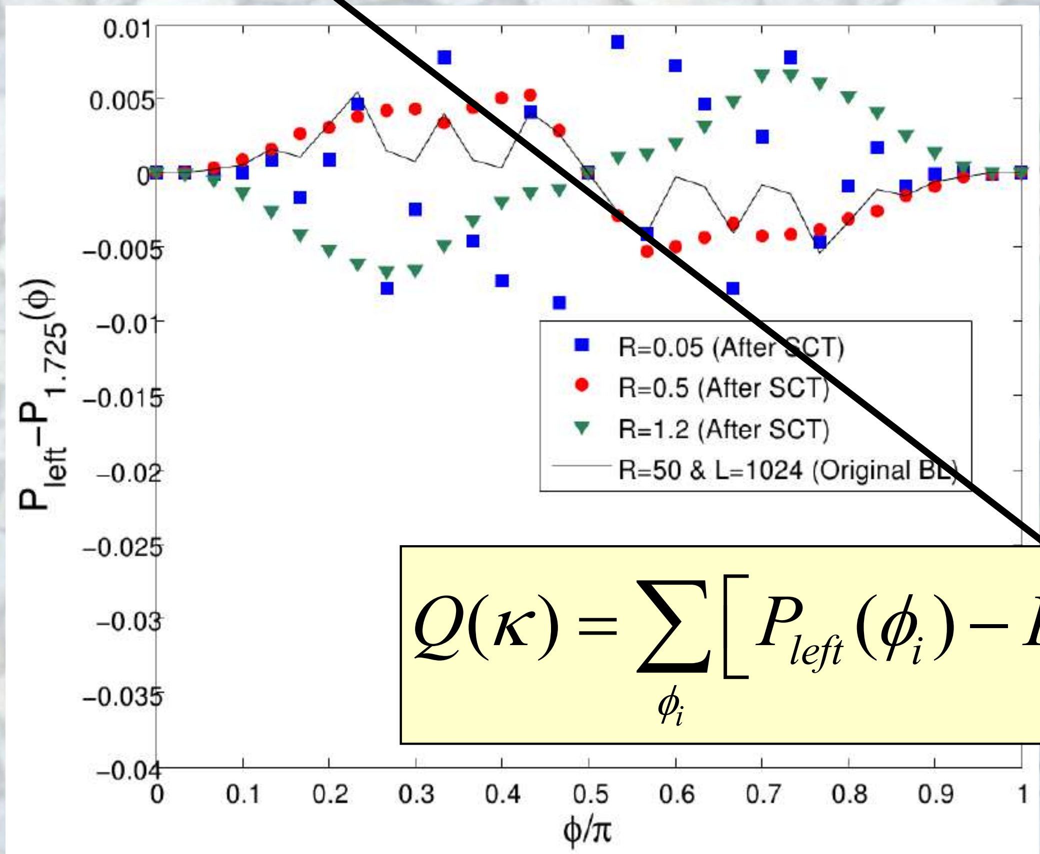


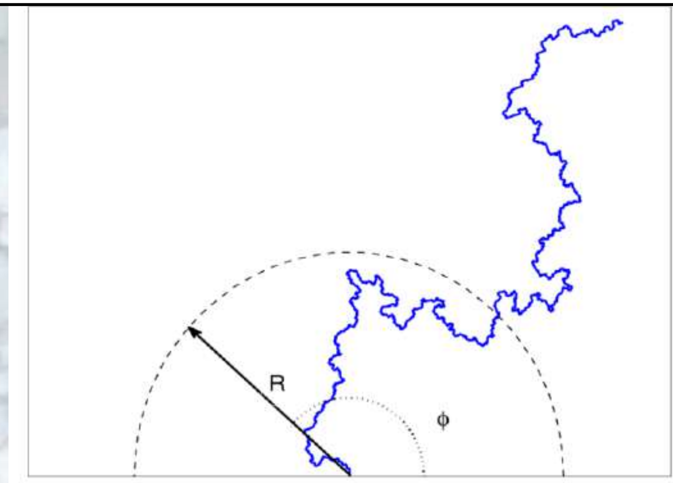
$$< \theta^2 > = a + \frac{\kappa}{4} \ln(L)$$



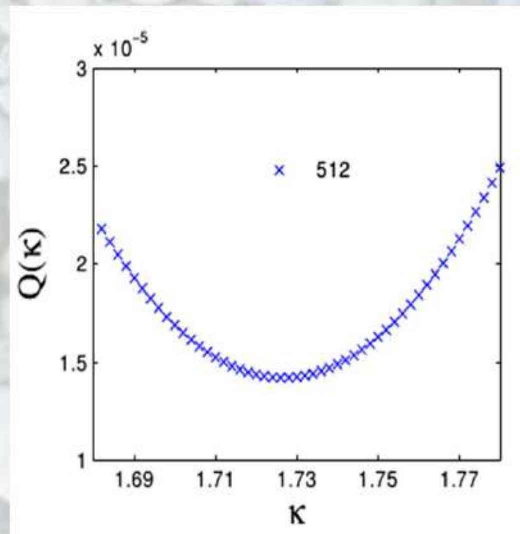
$$P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \cot(\phi)\,{}_2F_1\left(\frac{1}{2};\frac{4}{\kappa},\frac{3}{2};-\cot^2(\phi)\right)$$

Left-passage probability





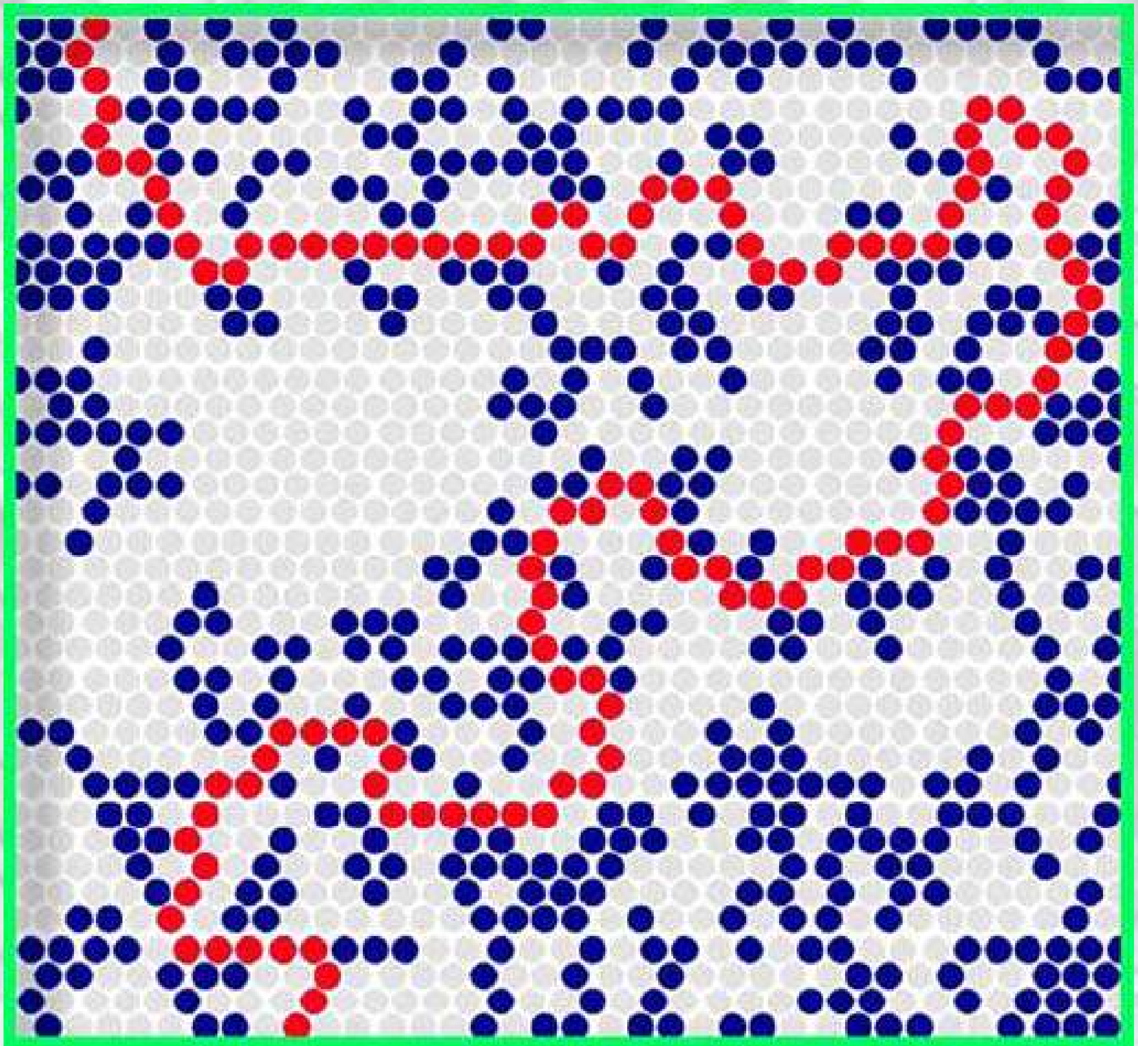
$$Q(\kappa) = \sum_{\phi_i}\left[P_{left}(\phi_i) - P_\kappa(\phi_i)\right]^2$$

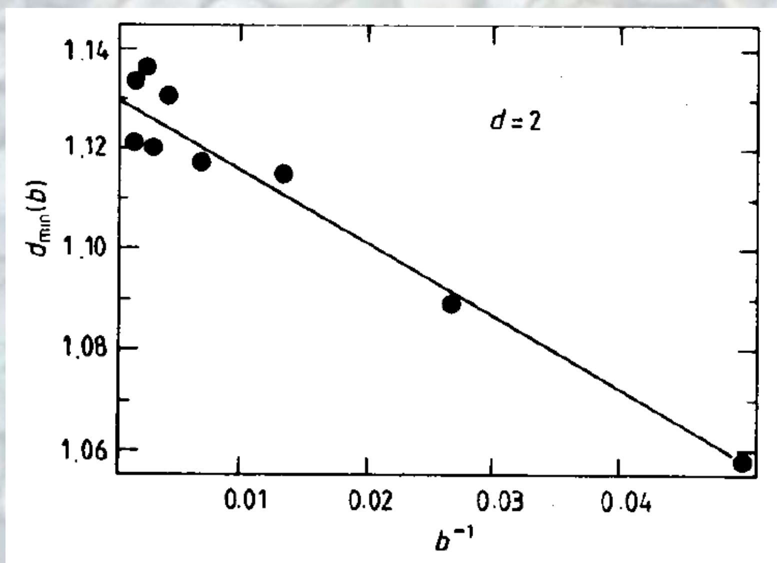


$$\kappa = 1.73 \pm 0.01$$

$$d_f = 1 + \frac{\kappa}{8} \approx 1.216$$

Shortest path on percolation cluster at p_c





H.J.H., H.E.Stanley, J.Phys.A 21, L829 (1988)

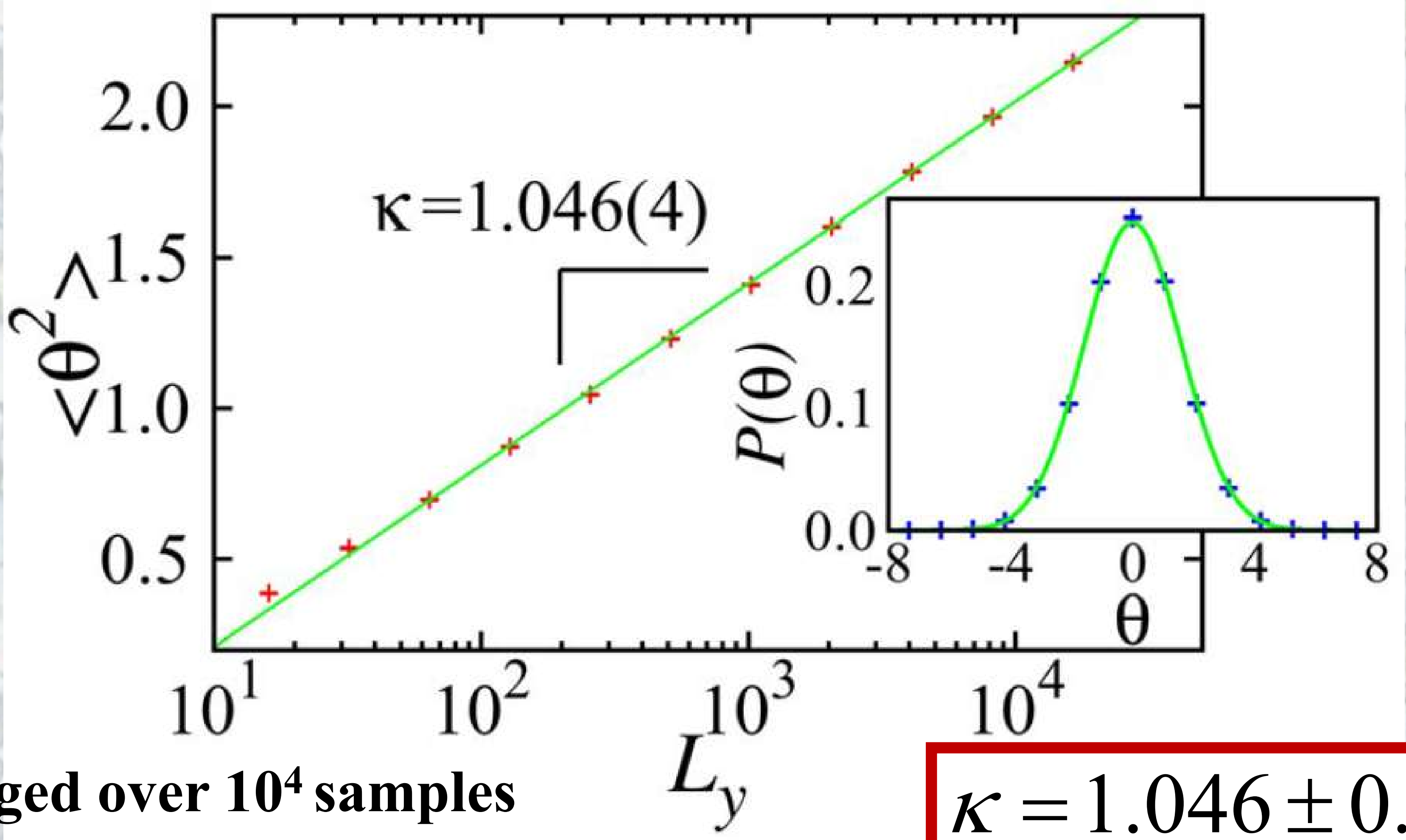
fractal dimension:

$d_f = 1.13077 \pm 0.00002$

Z.Zhou, J. Yang, Y. Deng, R.M. Ziff , Phys. Rev. E 86, 061101 (2012)

Shortest path

variance of the winding angle:



averaged over 10^4 samples

$$\kappa = 1.046 \pm 0.004$$

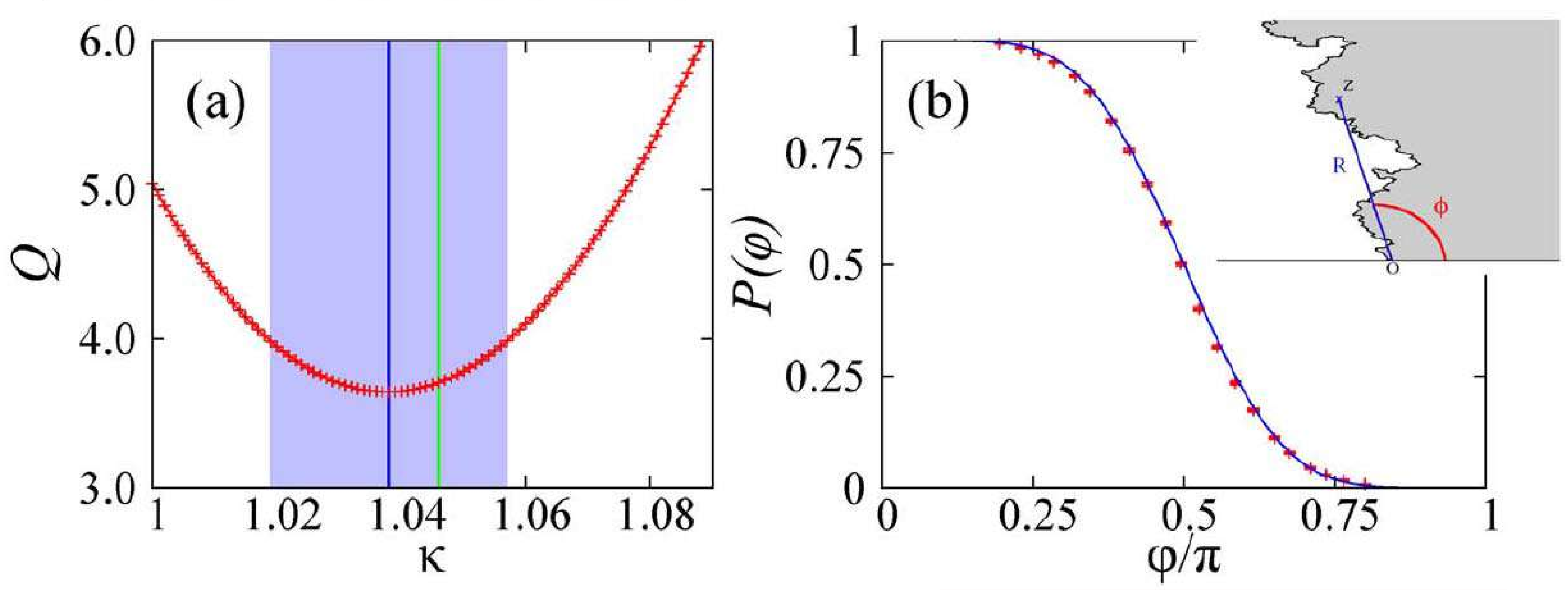
N. Posé, K.J. Schrenk, N.A.M. Araújo, H.J.H., Sci. Rep. 4, 5495 (2014)

Shortest path

left passage probability

$$P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma(4/\kappa)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \cot(\phi)\, {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -\cot(\phi)^2\right)$$

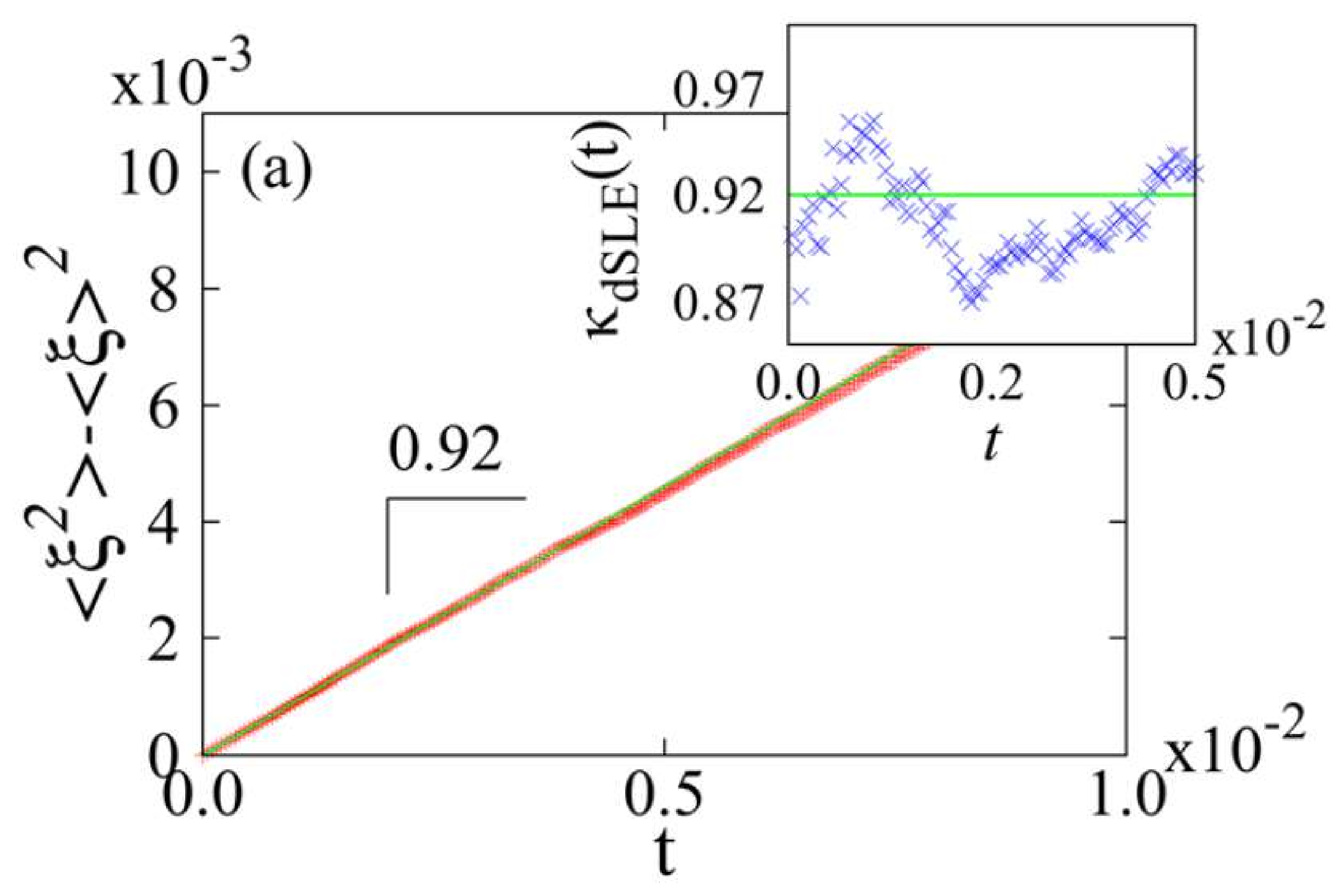
$$Q(\kappa) = \frac{1}{|S|} \sum_{z \in S} \frac{\left[P(z) - P_\kappa(\phi(z))\right]^2}{\Delta P(z)^2}$$



$\kappa = 1.038 \pm 0.019$

Shortest path

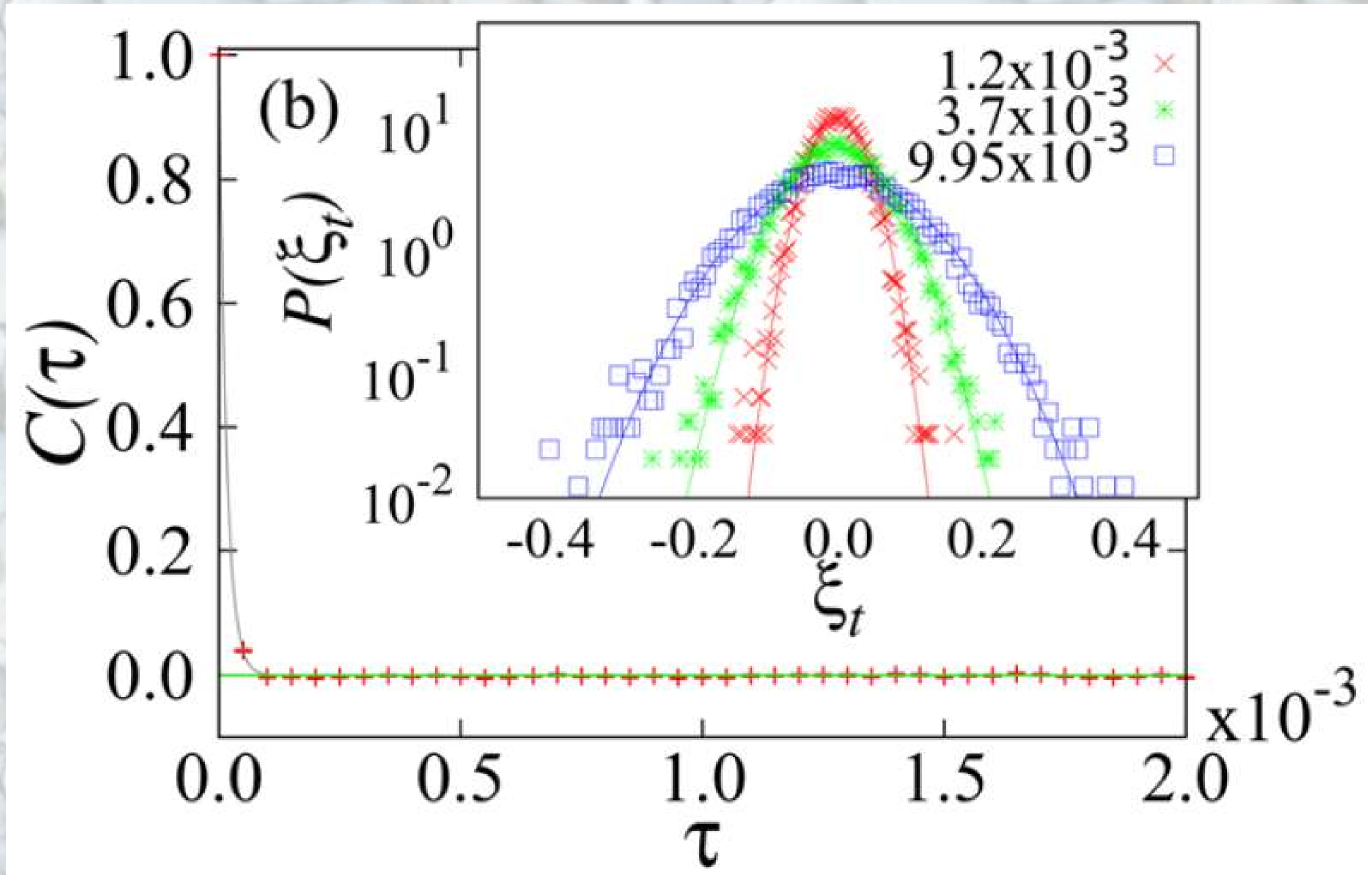
mean square deviation of the driving function against Loewner time



Shortest path

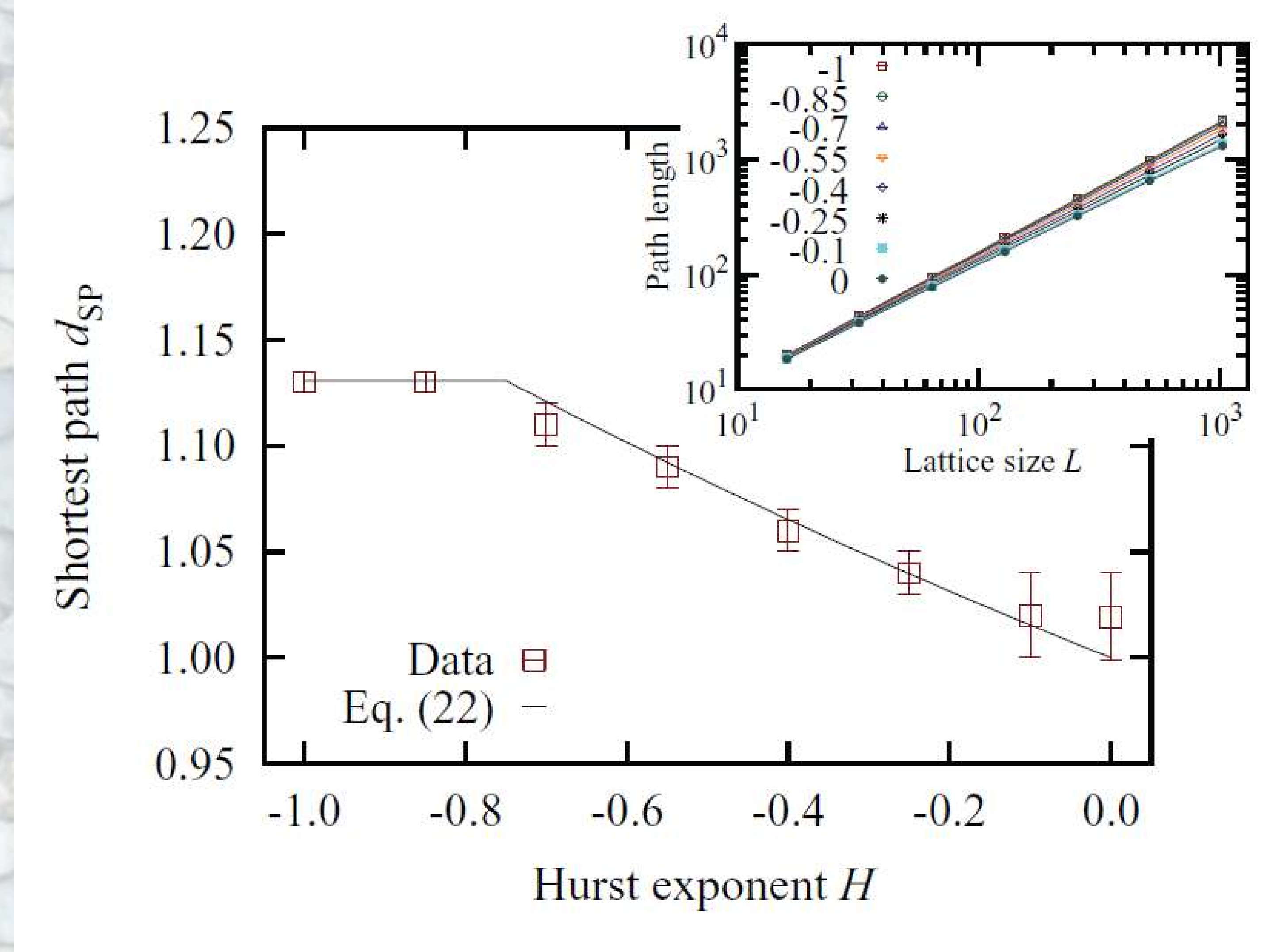
time correlations of the driving function

$$C(t,\tau) = \frac{\langle \delta\xi_{t+\tau}\delta\xi_t\rangle - \langle \delta\xi_{t+\tau}\rangle\langle \delta\xi_t\rangle}{\sqrt{(\langle \delta\xi^2_{t+\tau}\rangle - \langle \delta\xi_{t+\tau}\rangle^2)(\langle \delta\xi^2_t\rangle - \langle \delta\xi_t\rangle^2)}}.$$



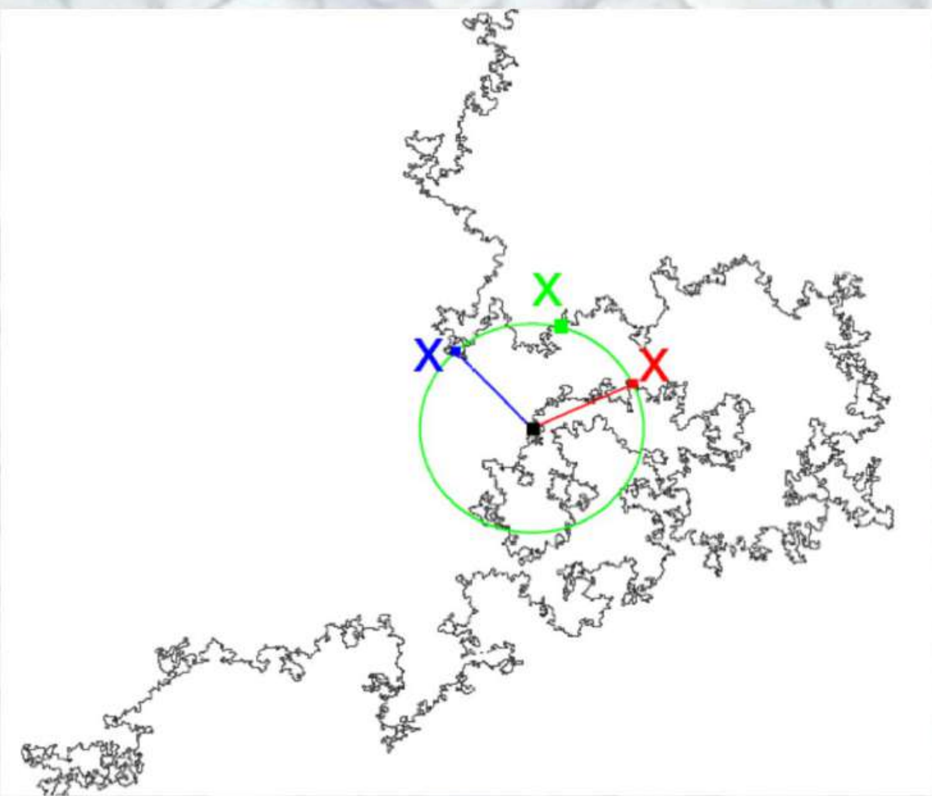
→ **Markov**

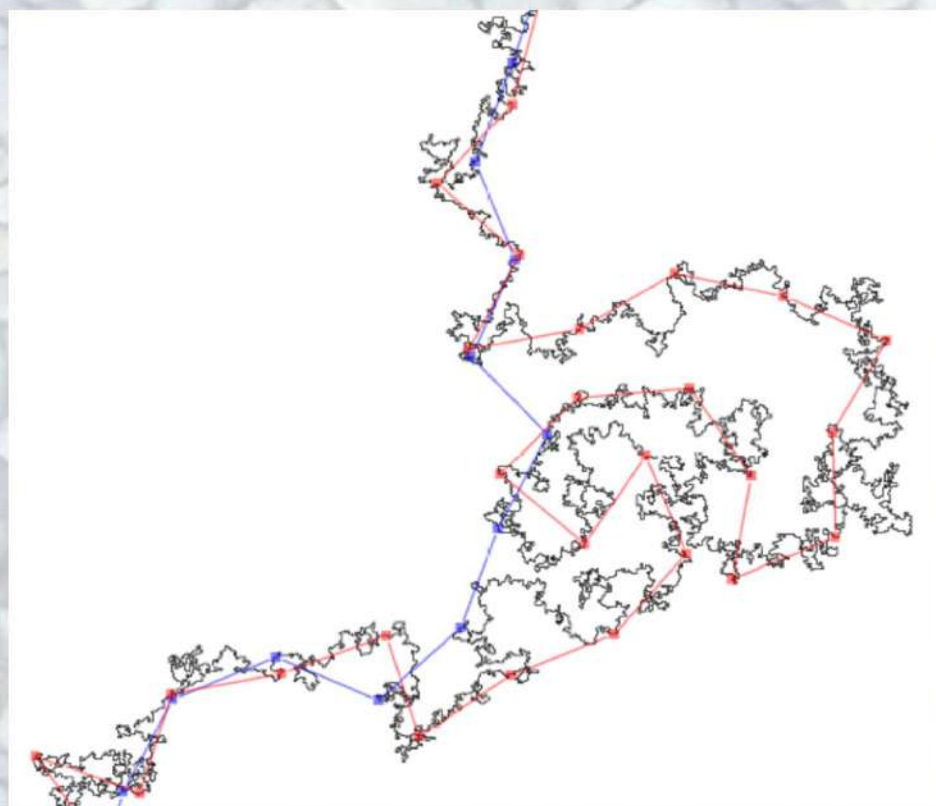
Shortest path on correlated landscapes



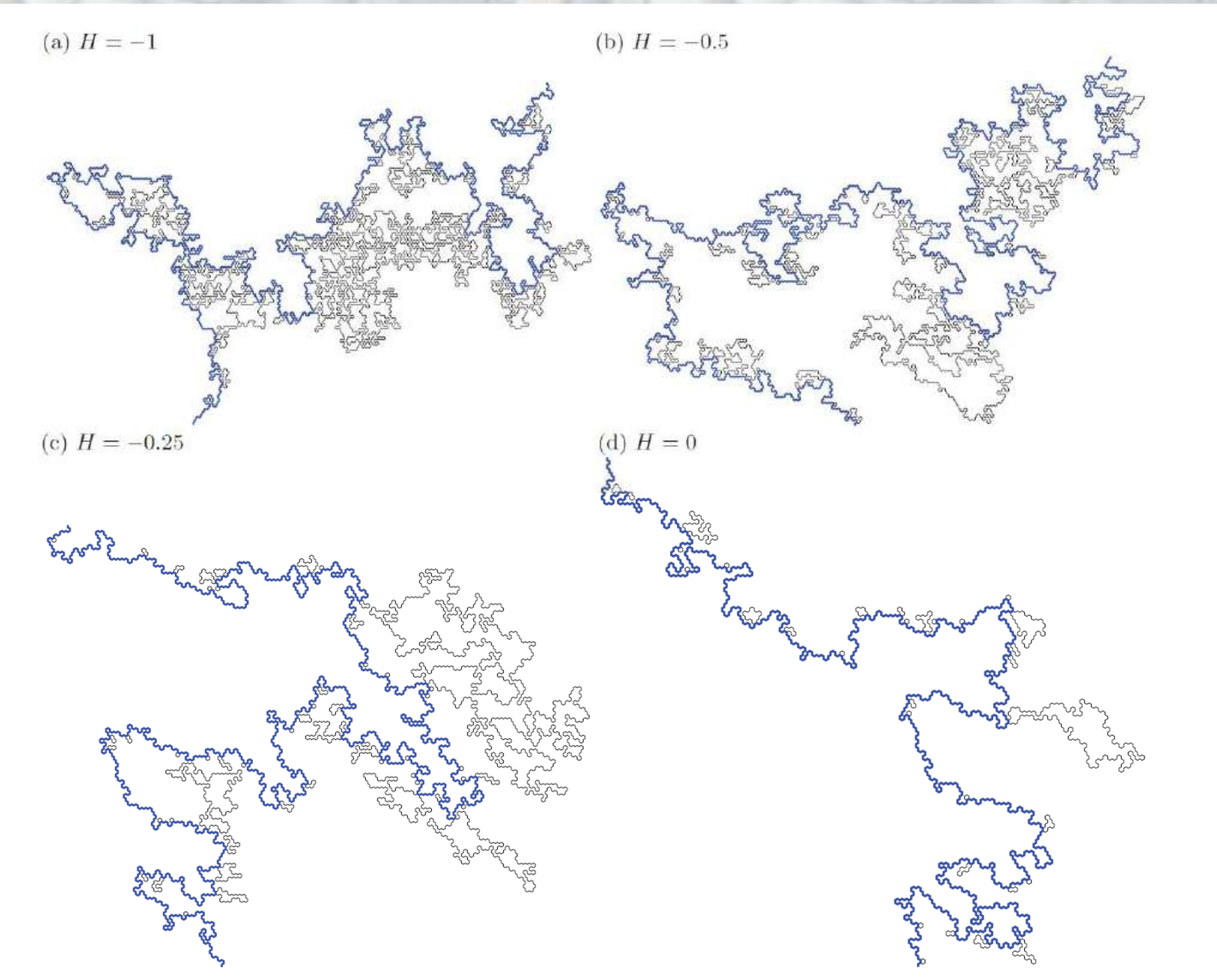
Are they SLE for all *H*?

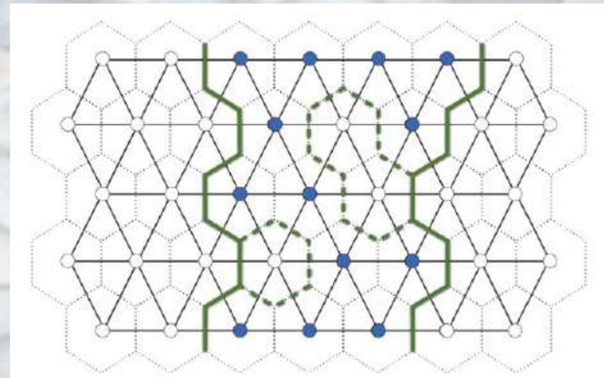
Complete and Accessible Perimeters





Complete and Accessible Perimeters





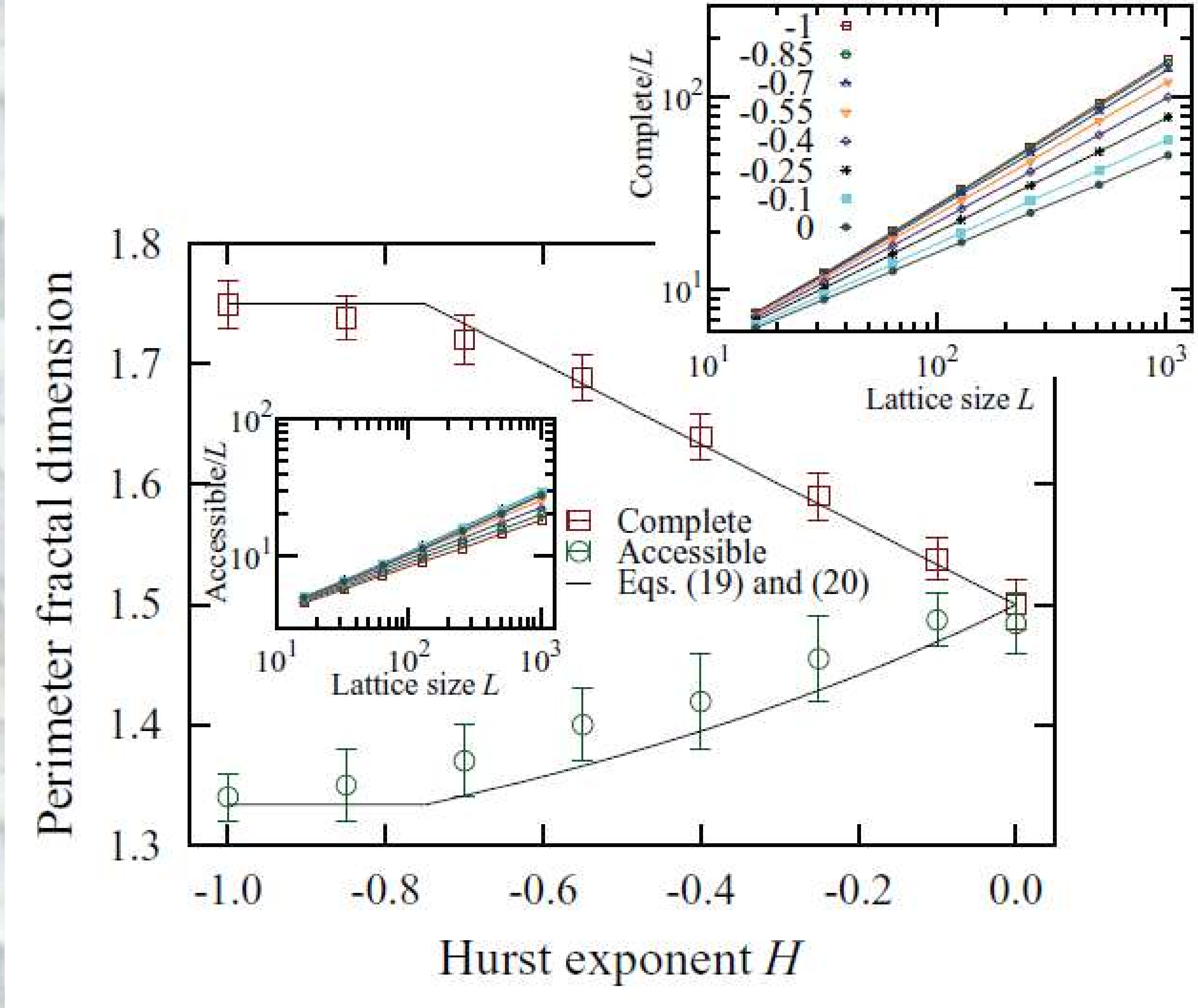
critical isoheight lines

Fractal Dimension of Perimeters

Fractal dimension of the complete and accessible perimeter of percolation on triangular lattice at $p_c = ½$ as function of the Hurst exponent of the random landscapes.

$$d_{CP} = \frac{3}{2} - \frac{H}{3}$$

$$d_{AP} = \frac{9-4H}{6-4H}$$

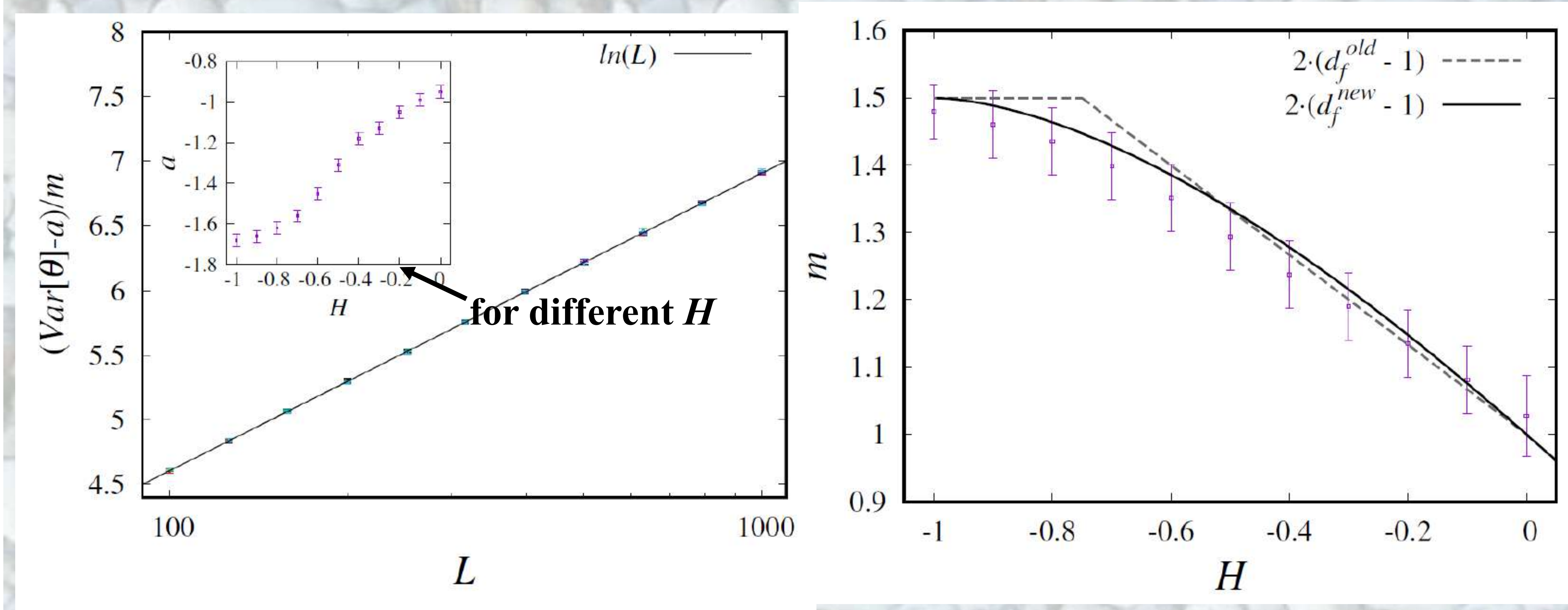


K.J. Schrenk, N. Posé, J.J. Krantz, L.V.M. van Kessenich, N.A.M. Araújo, H.J.H, Phys.Rev.E 88, 052102 (2013)

Complete perimeter on correlated landscapes

variance of the winding number

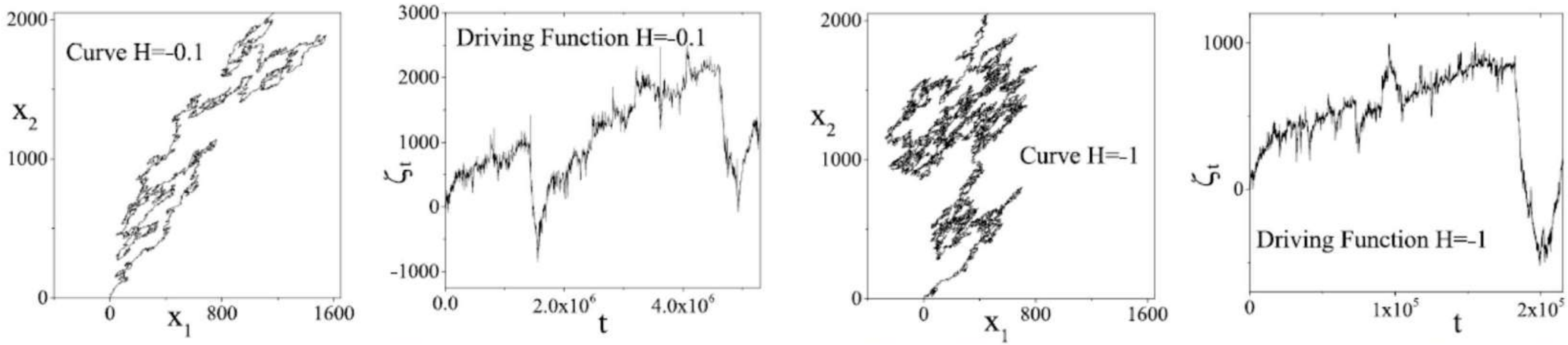
$$Var[\theta_L] = \langle \theta_L^2 \rangle - \langle \theta_L \rangle^2 = a + m \ln L \qquad m = \kappa/4$$



C.P. de Castro, M. Lukovic, G. Pompanin, R.F.S. Andrade, H.J.H.

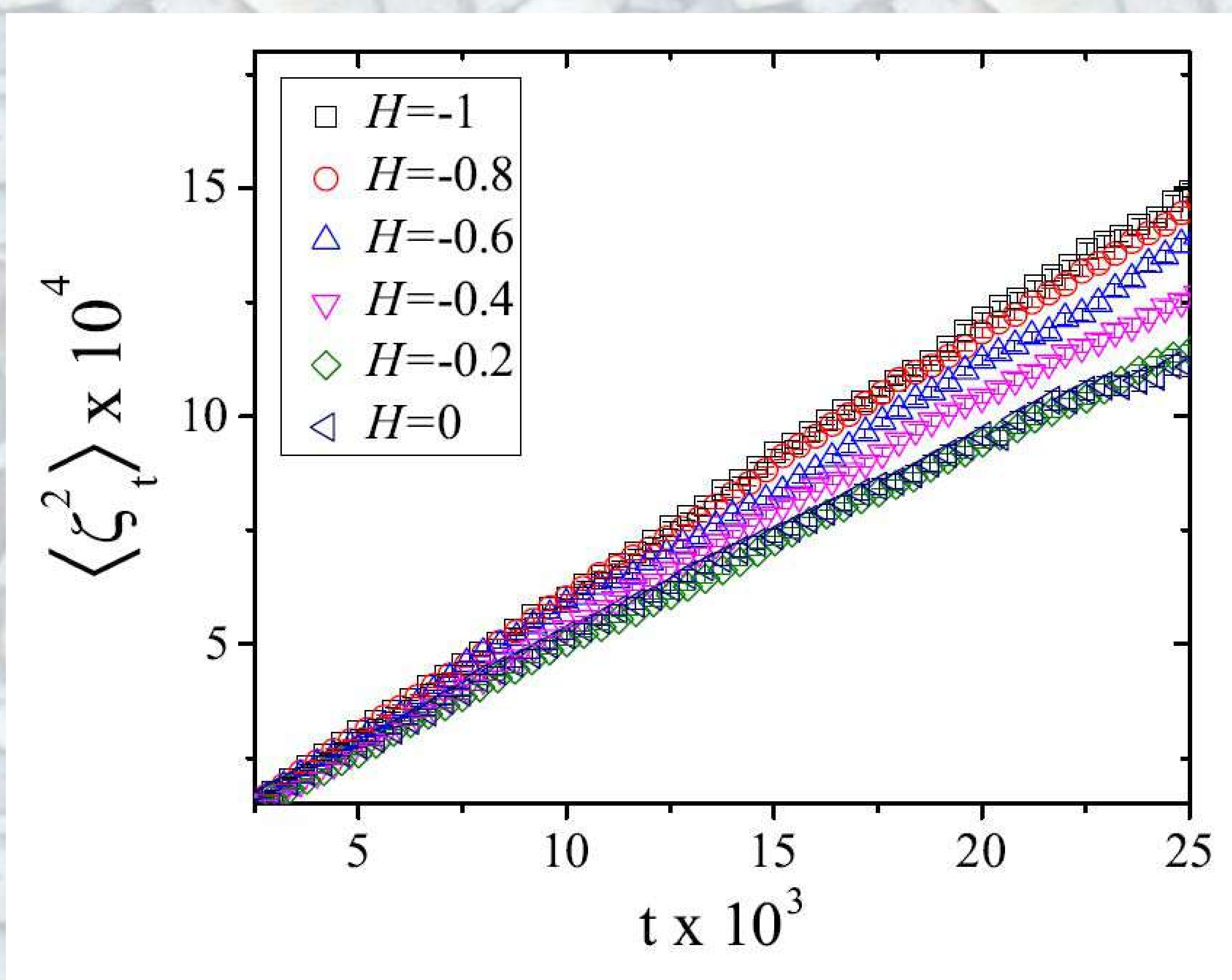
Sci. Rep. 7, 1961 (2017)

Complete perimeter on correlated landscapes



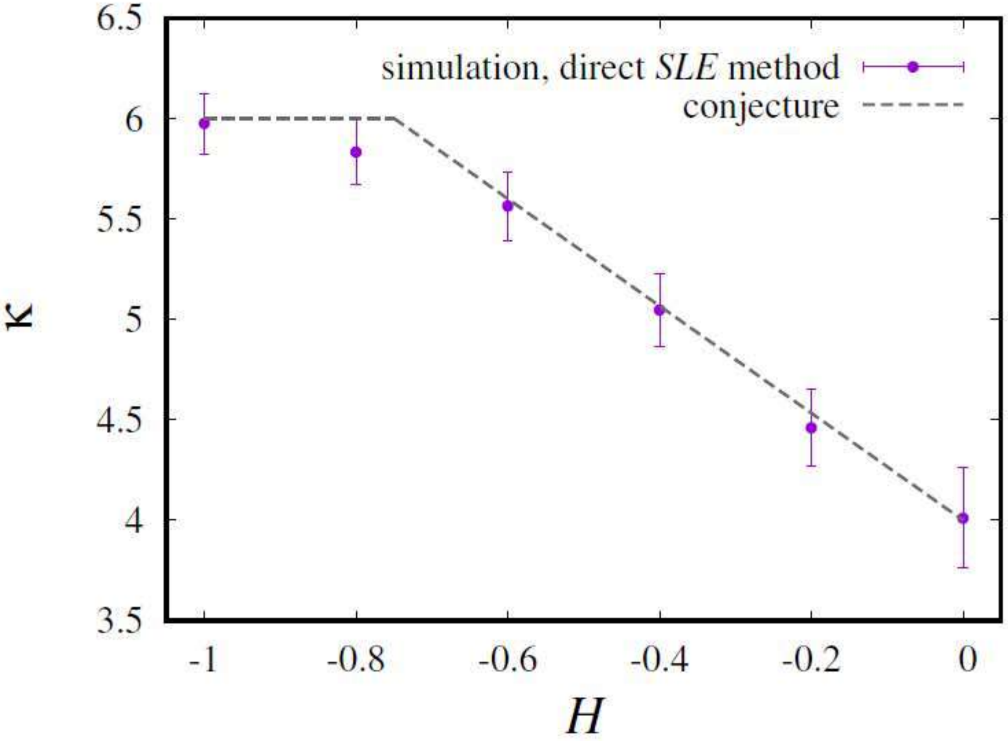
Complete perimeter on correlated landscapes

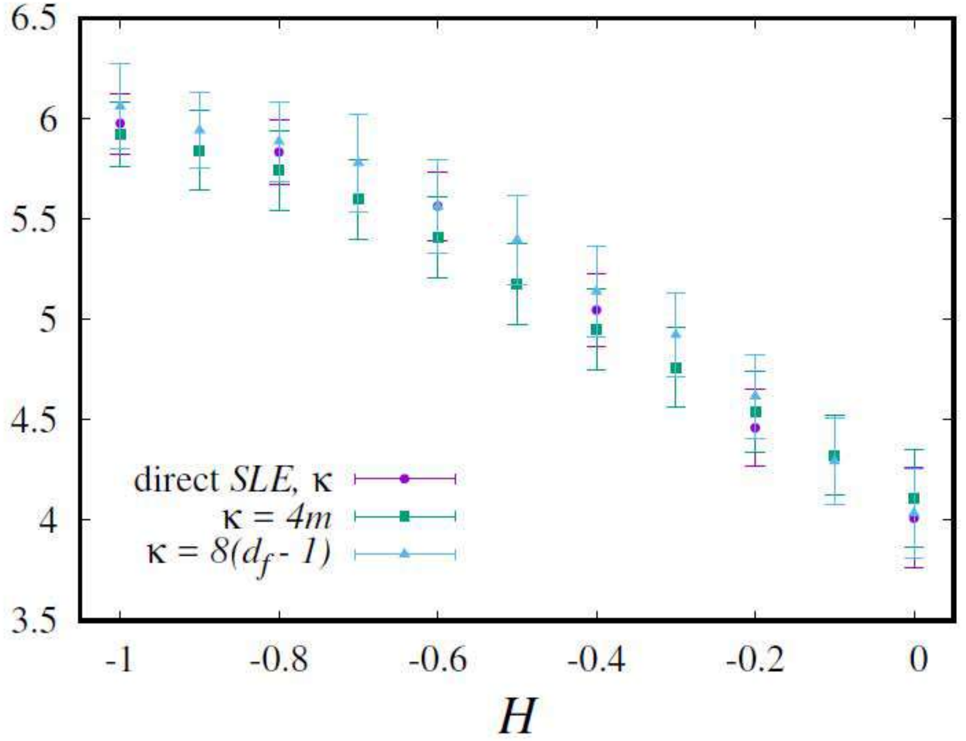
mean square displacement of the driving function



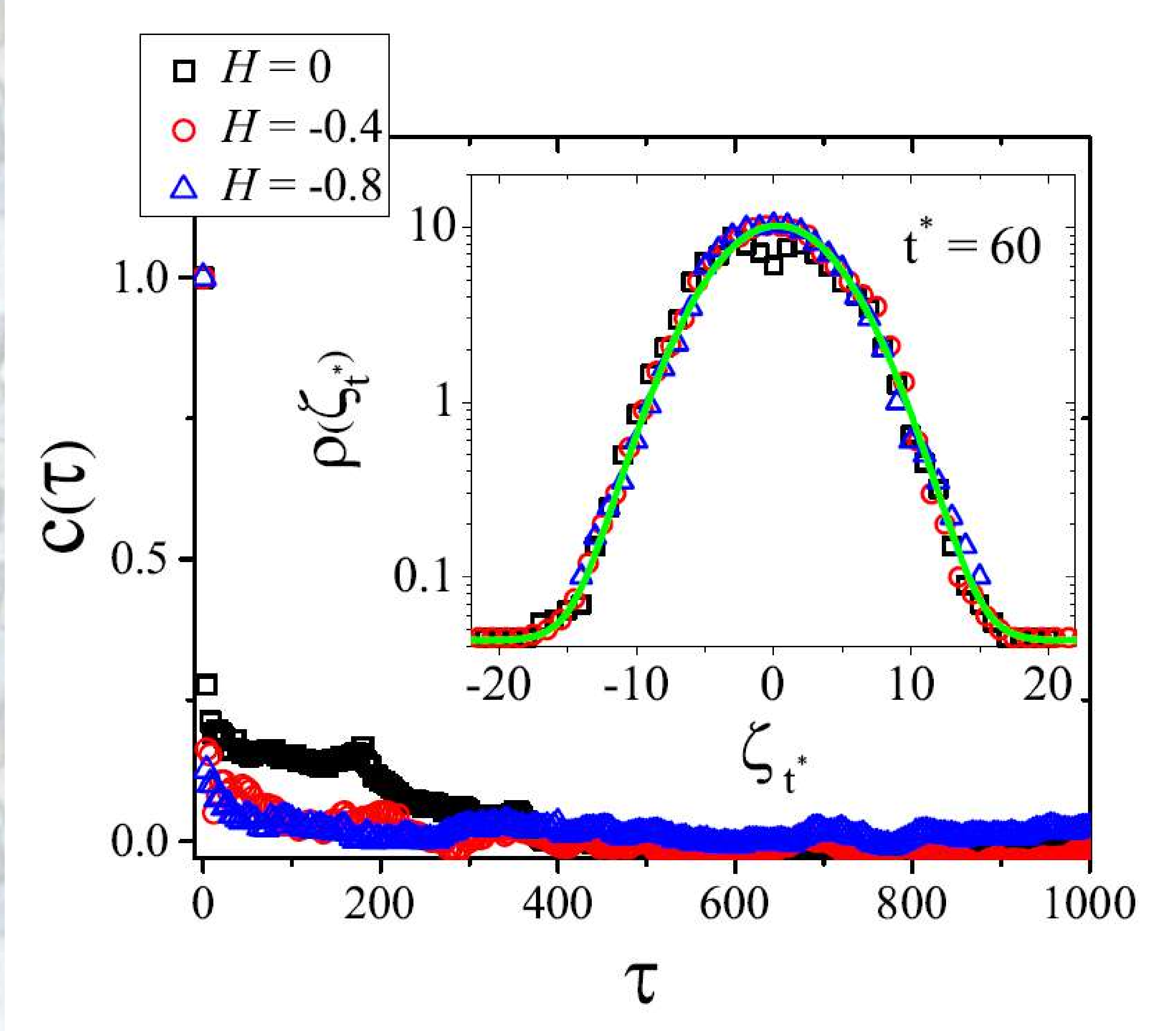
Complete perimeter on correlated landscapes

diffusion coefficient





Complete perimeter on correlated landscapes



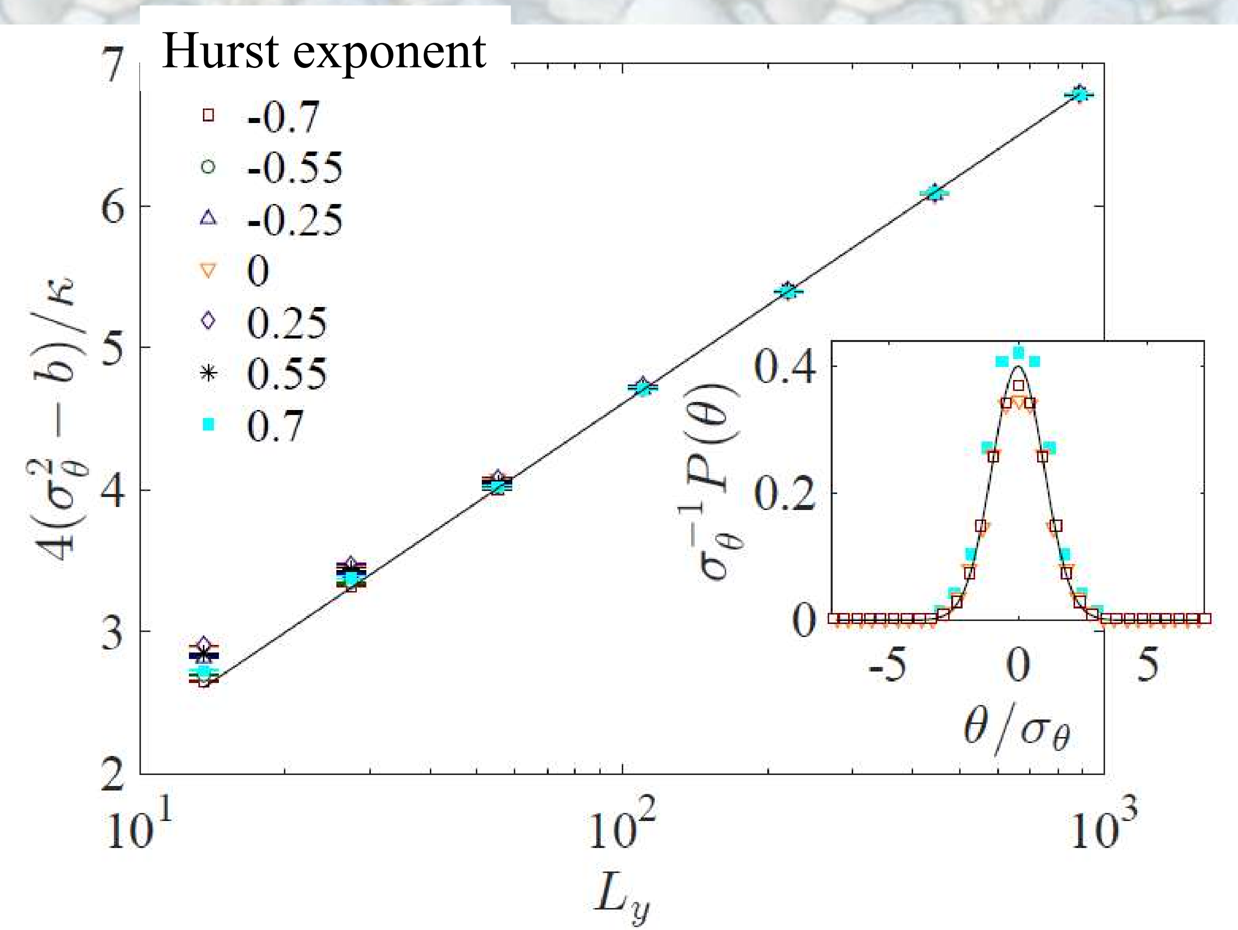
H = -0.8, -0.4 and 0

probability density distribution

time correlation

Accessible perimeter on correlated landscapes

rescaled variance of the winding number:

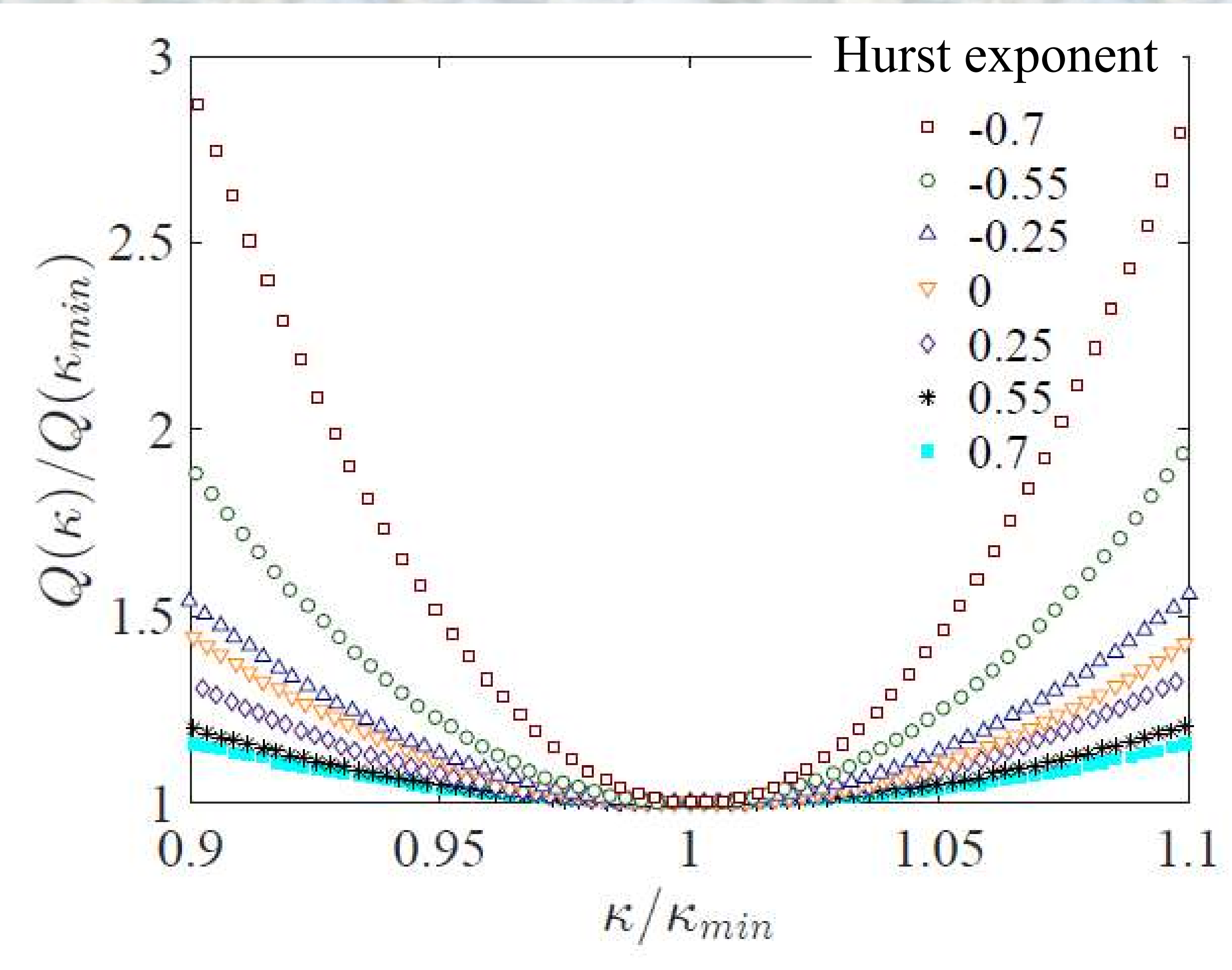


H	κ_θ
-1	2.66 ± 0.01
-0.85	2.76 ± 0.02
-0.7	2.94 ± 0.03
-0.55	3.14 ± 0.03
-0.4	3.32 ± 0.03
-0.25	3.45 ± 0.09
-0.1	3.49 ± 0.04
0	3.44 ± 0.18
0.1	3.44 ± 0.27
0.25	3.07 ± 0.07
0.55	2.22 ± 0.08
0.7	1.65 ± 0.16
0.95	0.62 ± 0.31

N. Posé, K.J. Schrenk, N.A.M. Araújo, H.J.H., IJMPC

Accessible perimeter on correlated landscapes

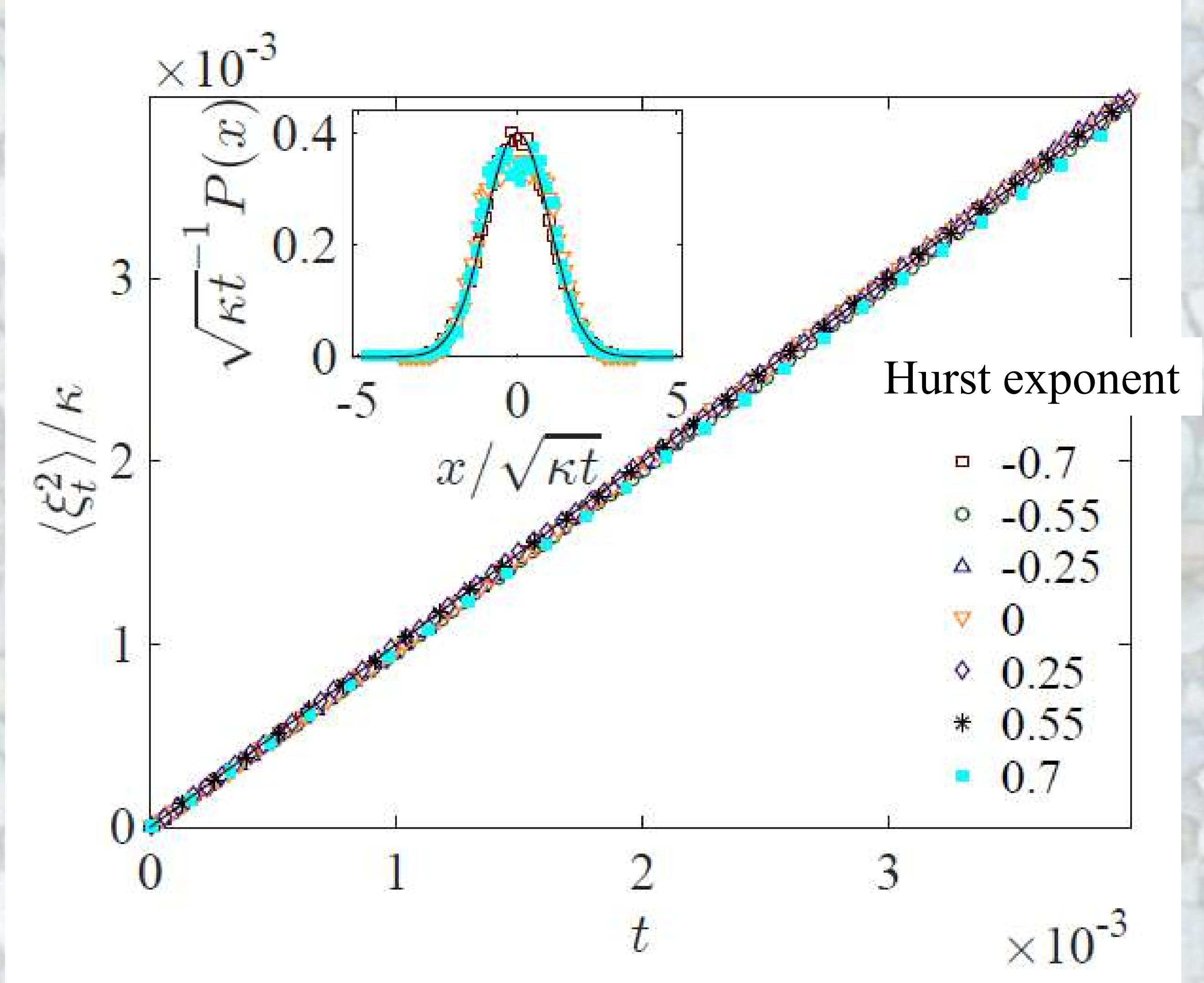
left passage probability



H	κ_{LPP}
-1	2.69 ± 0.08
-0.85	2.80 ± 0.07
-0.7	2.95 ± 0.11
-0.55	3.13 ± 0.15
-0.4	3.27 ± 0.17
-0.25	3.40 ± 0.21
-0.1	3.50 ± 0.24
0	3.52 ± 0.25
0.1	3.56 ± 0.26
0.25	3.59 ± 0.29
0.55	3.62 ± 0.36
0.7	3.62 ± 0.41
0.95	3.63 ± 0.50

Accessible perimeter on correlated landscapes

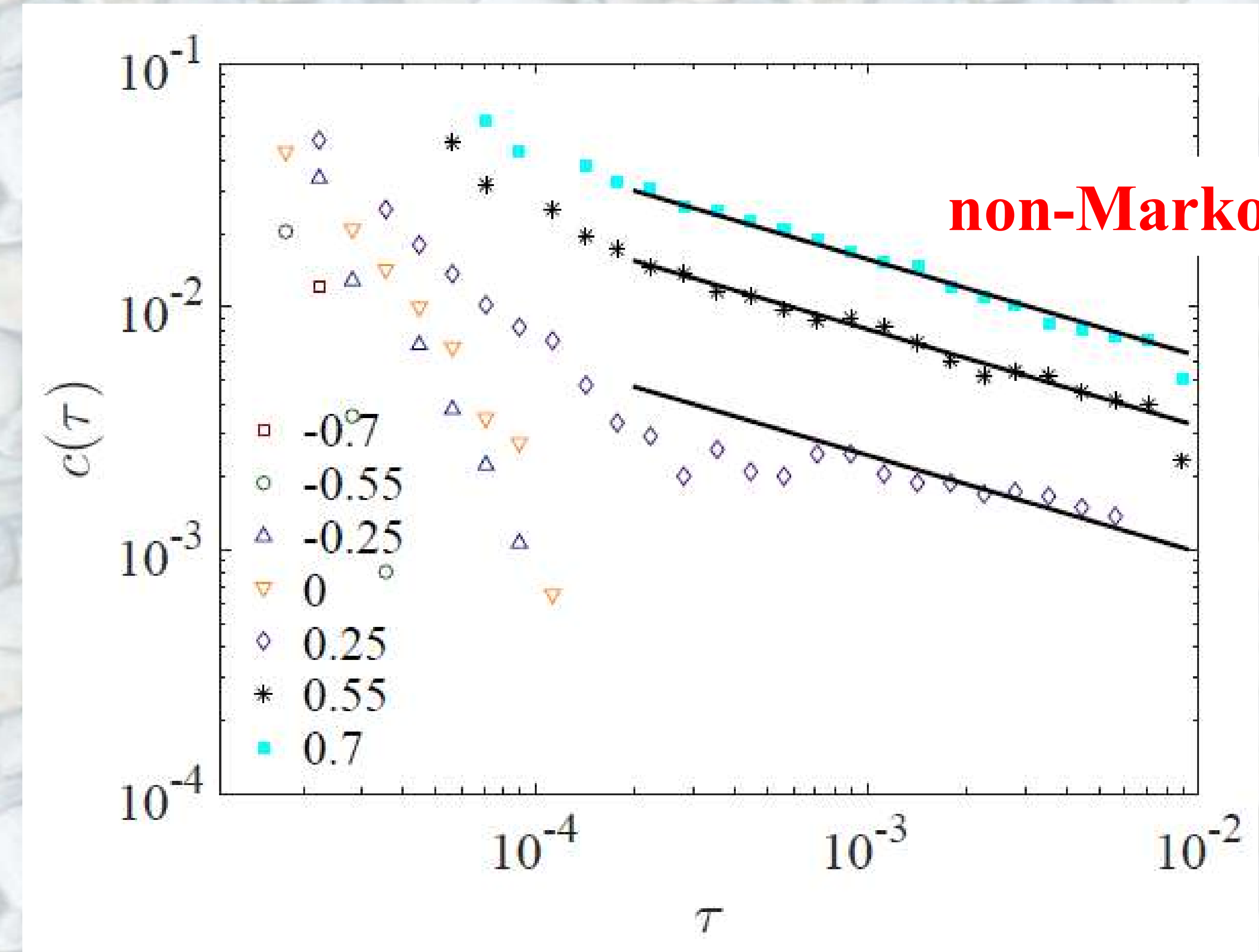
variance of the driving function against Loewner time



H	κ_{dSLE}
-1	2.66 ± 0.12
-0.85	2.79 ± 0.10
-0.7	2.97 ± 0.17
-0.55	3.29 ± 0.32
-0.4	3.46 ± 0.26
-0.25	3.67 ± 0.26
-0.1	3.84 ± 0.20
0	3.98 ± 0.19
0.1	4.07 ± 0.13
0.25	4.26 ± 0.13
0.55	4.84 ± 0.29
0.7	5.35 ± 0.25
0.95	5.97 ± 0.51

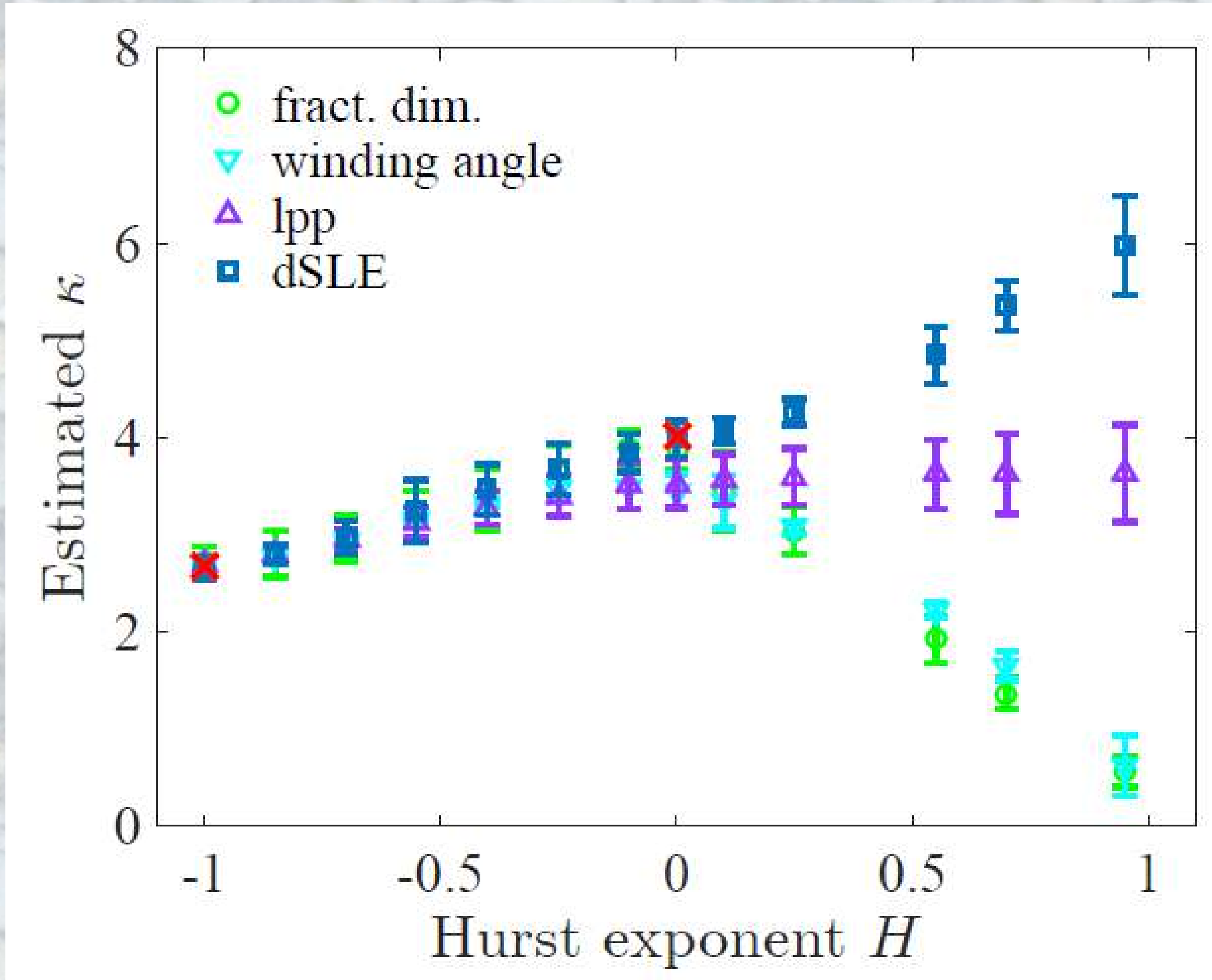
Accessible perimeter on correlated landscapes

correlations in time for different Hurst exponents



non-Markovian for $H > 0$

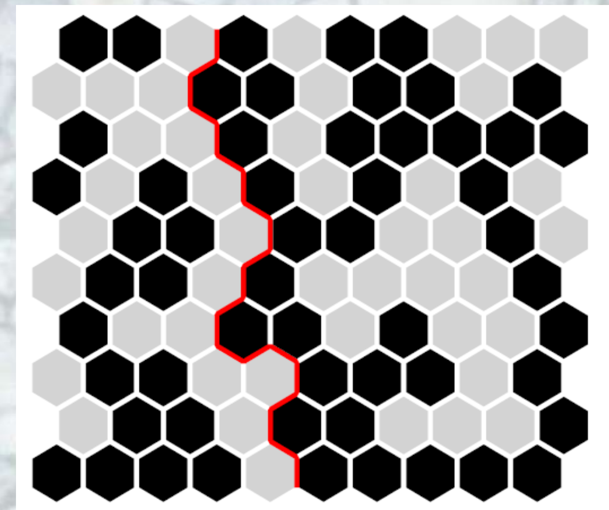
Accessible perimeter on correlated landscapes

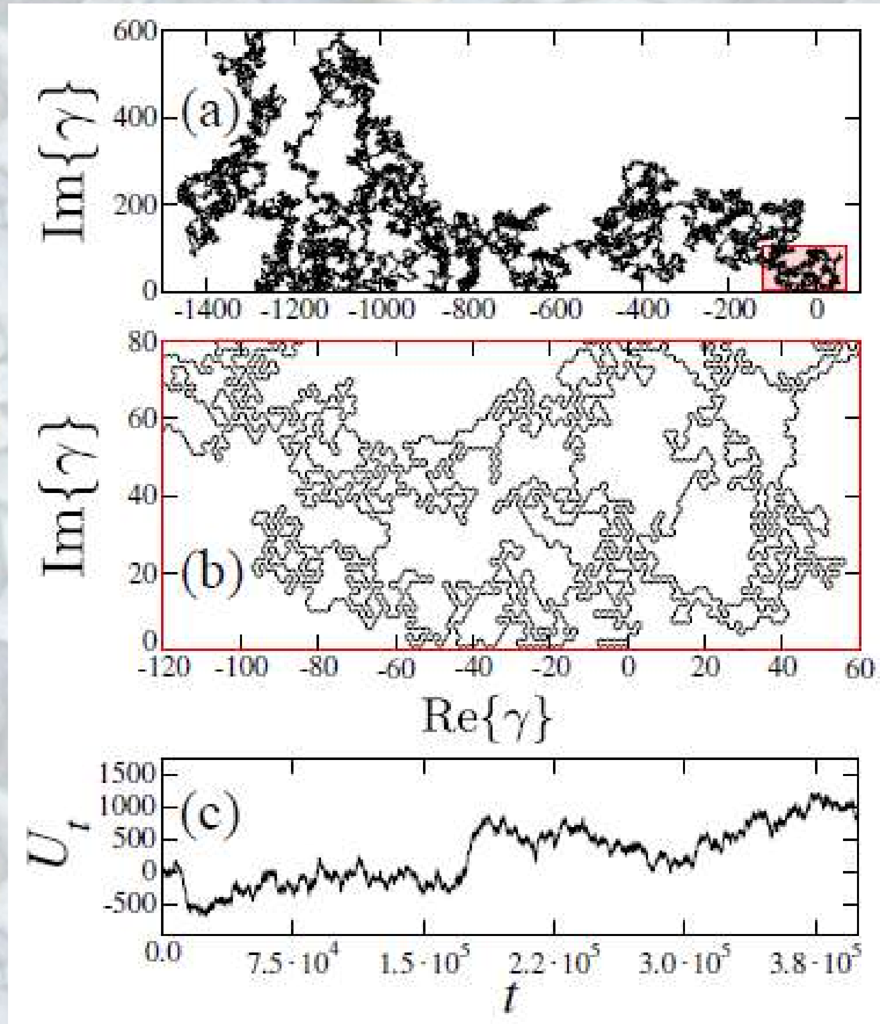


Beyond conformal invariance

perimeter of percolation cluster

at $p_c = ½$





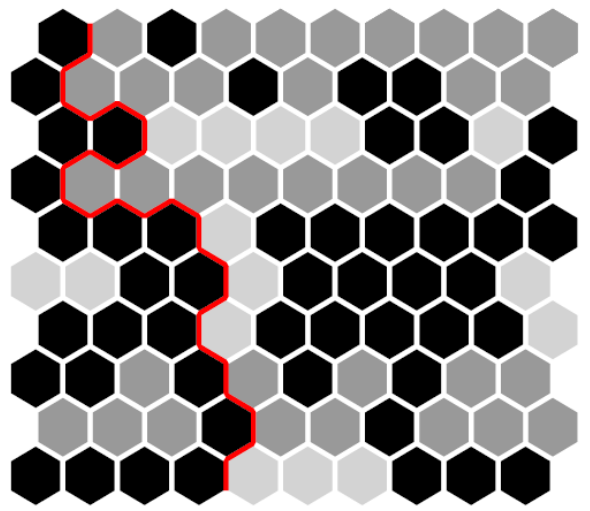
H.F. Crededio, A.A. Moreira, H.J.H., J.S. Andrade Jr.,

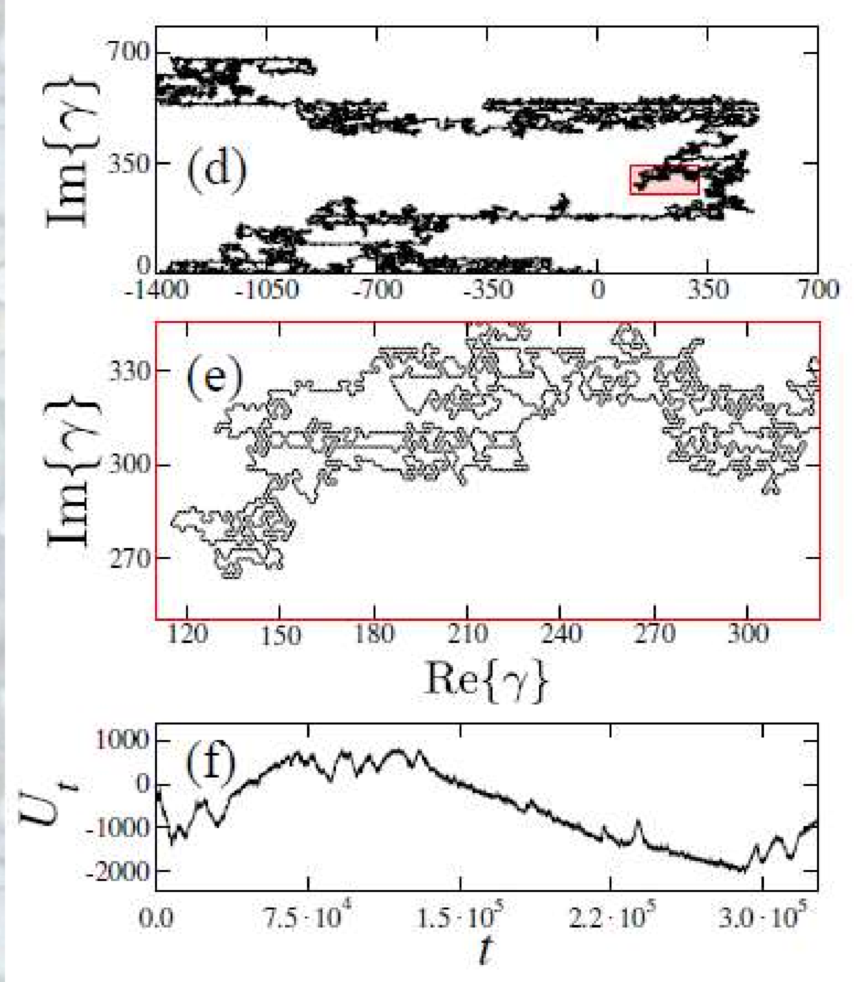
Phys. Rev. E 93, 042124 (2016)

Multi-layered percolation

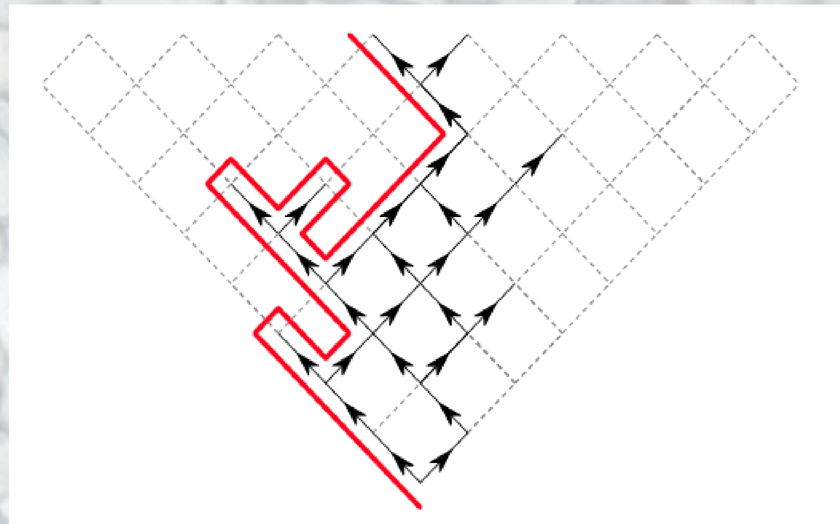
probability to occupy site: $p \pm \Delta$
where for each line the sign
is chosen randomly

Δ is the degree of anisotropy

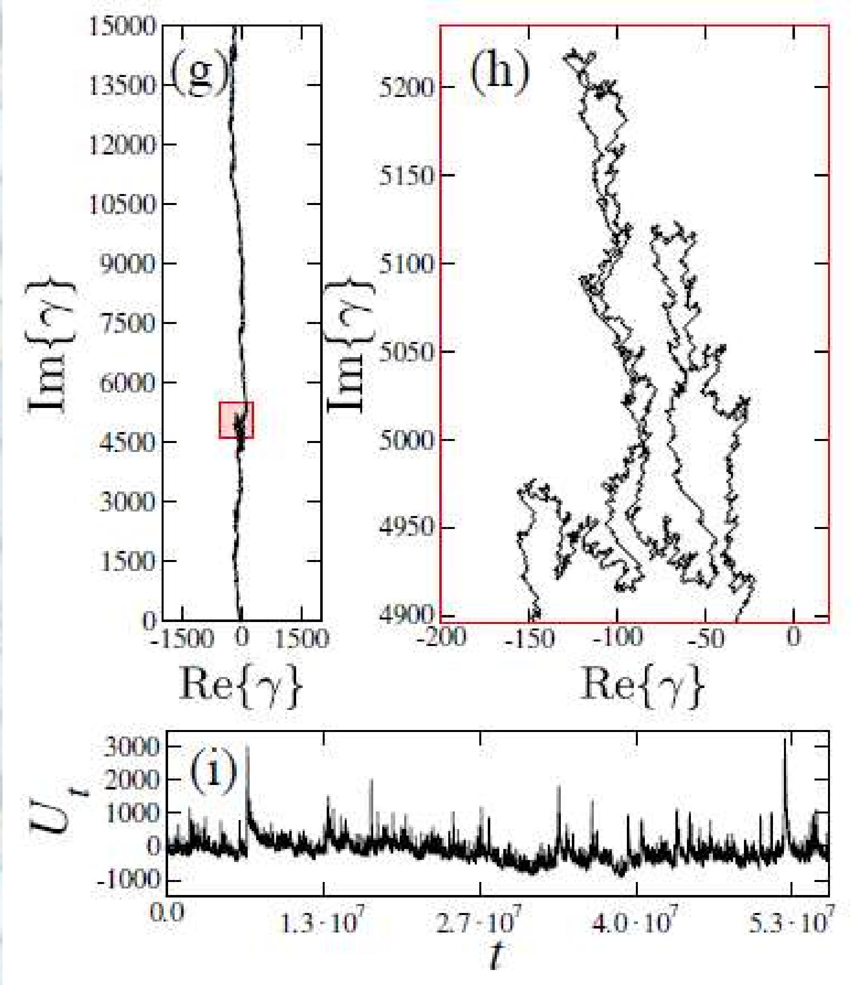




Directed percolation

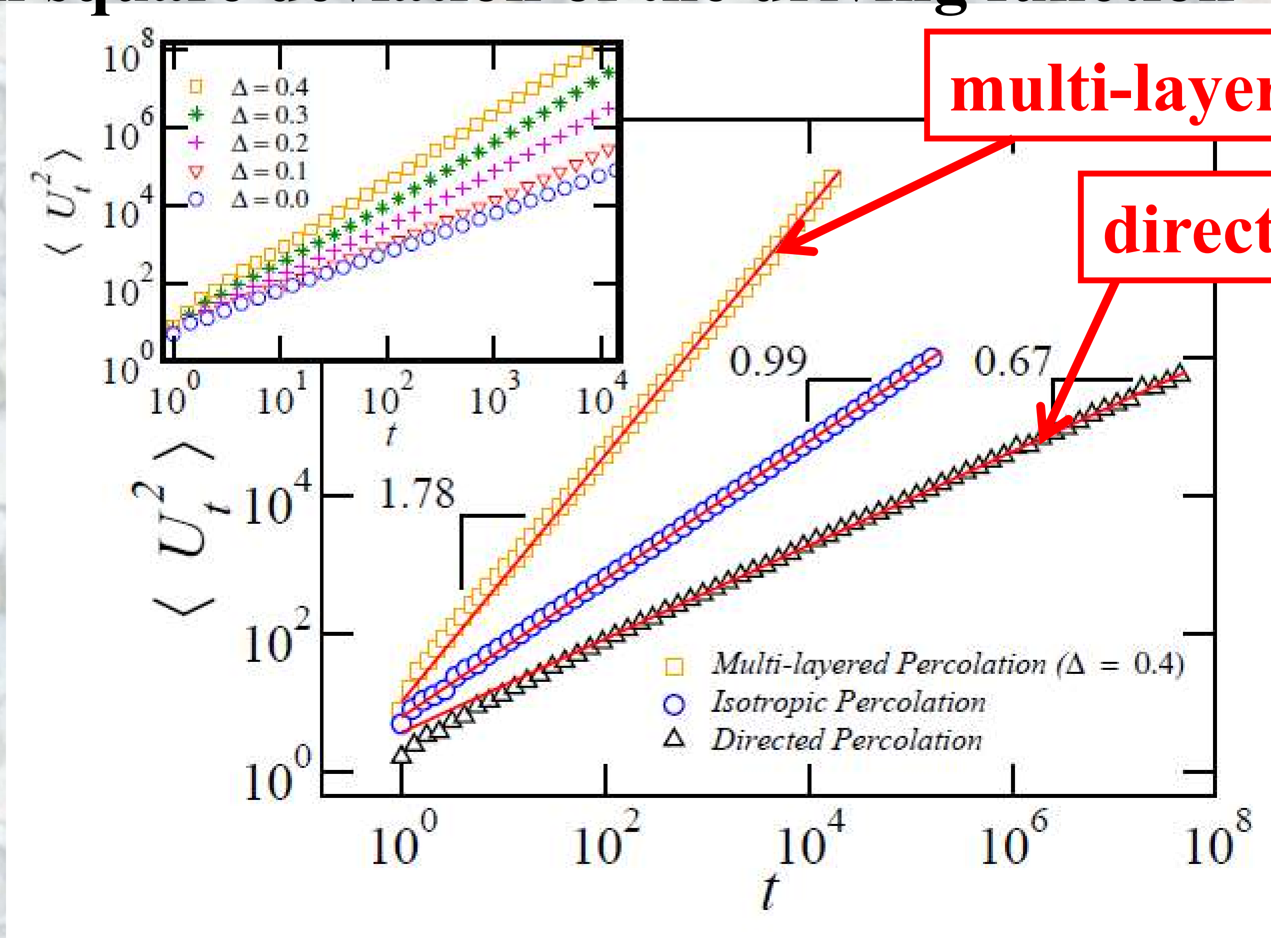


p_c = **0.644700185(5) (Jensen,99)**



Anisotropic models

mean square deviation of the driving function



$$\langle U_t^2 \rangle = b\, t^{\alpha}$$

averaged over 10^4 traces of length 10^5

H.F. Credidio, A.A. Moreira, H.J.H., J.S. Andrade Jr., Phys. Rev. E 93, 042124 (2016)

Anisotropic models

Inverse operation: start with a (discretized) driving function U_t obtained from fractional Brownian motion with Hurst exponent H i.e. following $\langle U_t^2 \rangle = b\, t^{2H}$

Then obtain a trace in complex plane from $\gamma_i = g_0 \circ g_1 \circ \ldots \circ g_i(0),$

with $g_i(z) = i\sqrt{4(t_i - t_{i-1})^2 - z^2} + (U_{t_i} - U_{t_{i-1}}).$

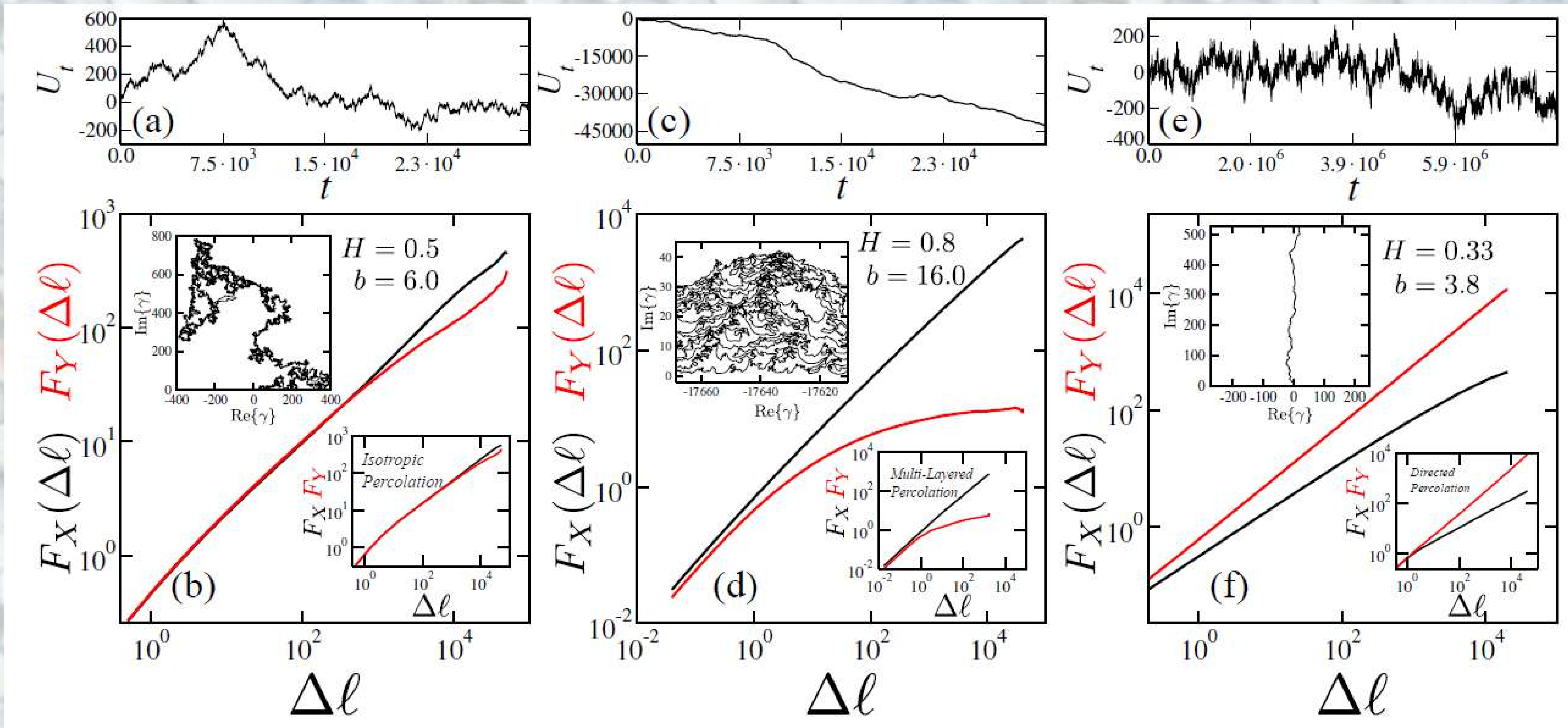
and measure its anisotropy with:

$$F_X(i\Delta l) = \sqrt{\frac{1}{M-i}\sum_{j=0}^{M-i}\left[\mathrm{Re}\{\gamma(l_{j+1})\} - \mathrm{Re}\{\gamma(l_j)\}\right]^2}$$

Anisotropic models

mean square deviation in X and Y direction of SLE traces driven by time series following anomalous diffusion (fBm):

uncorrelated **persistent** **anti-persistent**



The Saga of Explosive Percolation

Dimitris Achlioptas

Raissa D'Souza

Joel Spencer

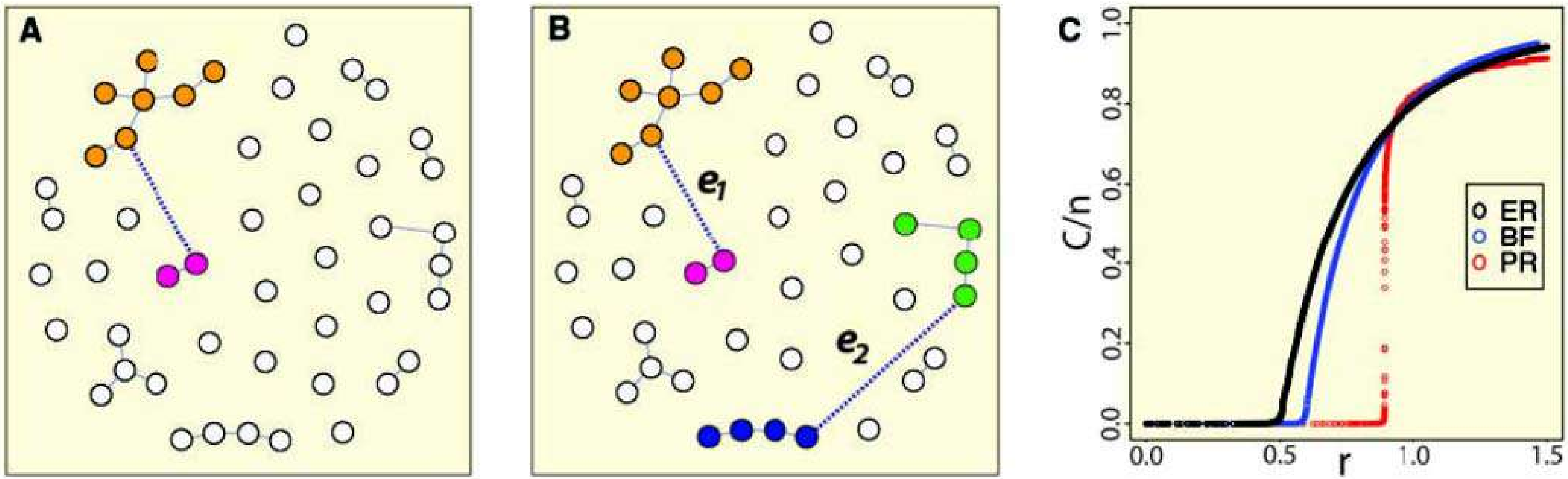
D. Achlioptas, R. M. D'Souza and J. Spencer, Science 323, 1453 (2009)

Product Rule (PR)

- **Consider a fully connected graph.**
- **Select randomly two bonds and occupy the one which creates the smaller cluster.**

classical percolation

product rule



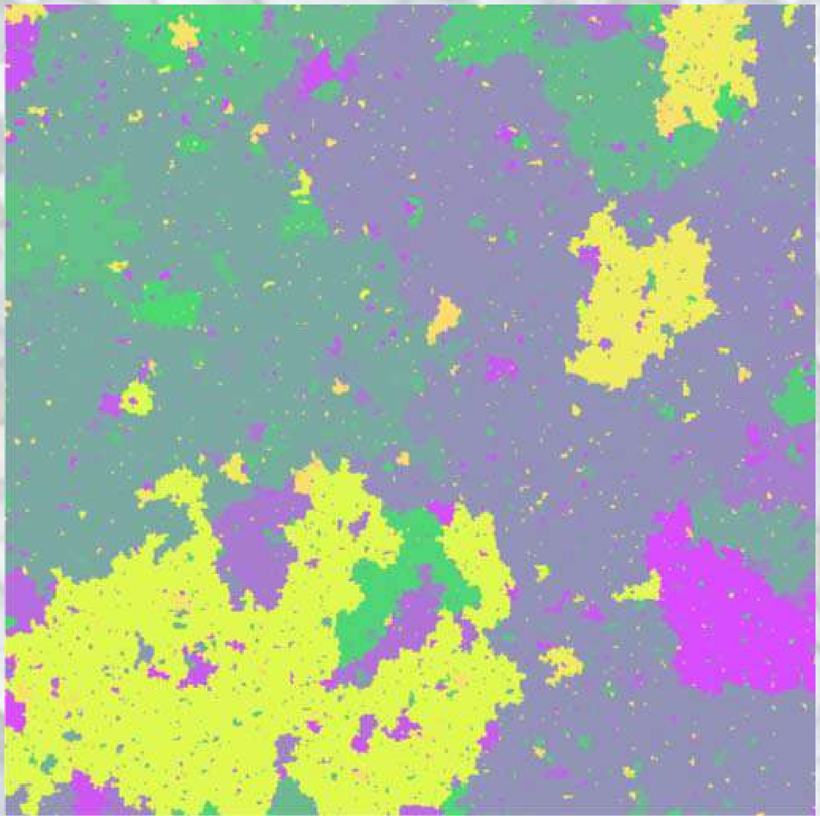
D. Achlioptas, R. M. D'Souza and J. Spencer, Science 323, 1453 (2009)

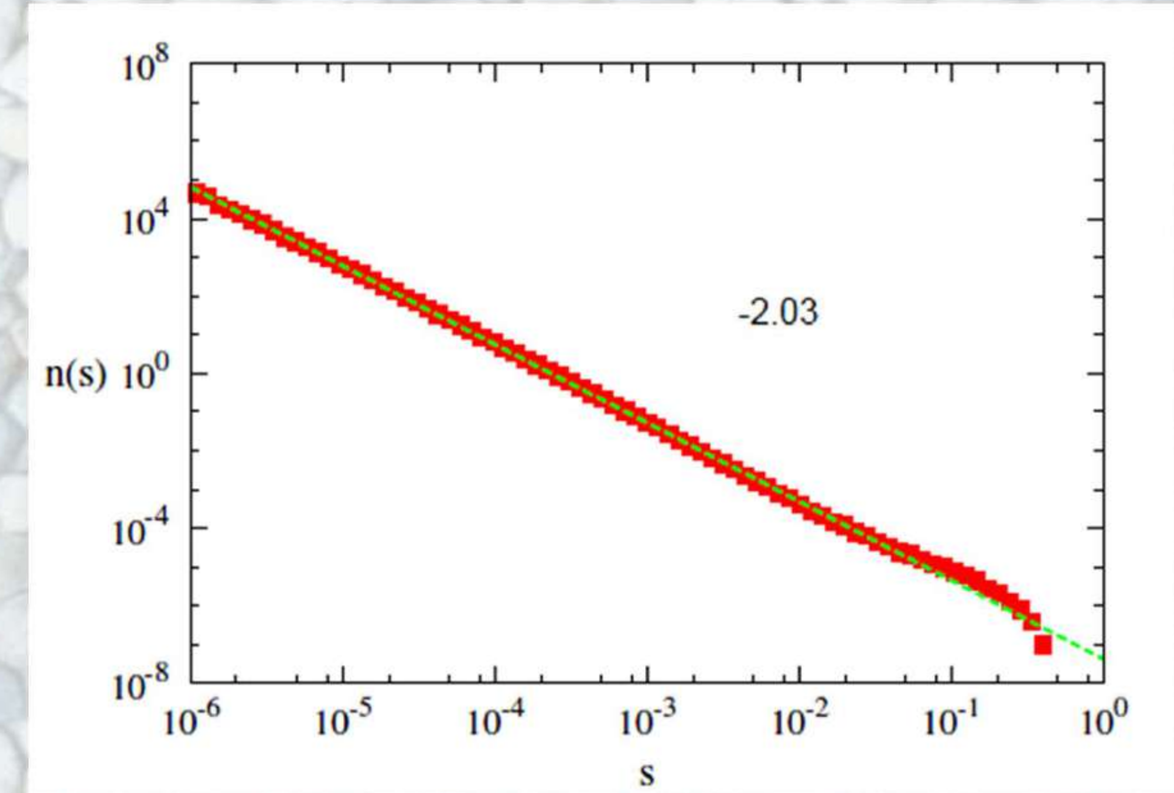
Product Rule (PR)

cluster size distribution n_s

on the square lattice:

$$n_s \propto s^{-\tau}$$





Y. S. Cho et al., Phys. Rev. E 82, 042102 (2010)

However, ...

Transition continuous in thermodynamic limit

J. Nagler, A. Levina and T. Timme, Nature Phys. 7, 2645 (2010)

O. Riordan and L. Warnke, Science, 333, 322 (2011)

R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett., 105, 255701 (2010)

But what happens in finite dimension ??

Best-of-*m* Model

José Soares Andrade Jr.

- Select randomly *m* bonds and occupy the one which creates the smaller cluster

This is a straightforward generalization of the Product Rule which corresponds to $m = 2$.
$m = 1$ is classical percolation.

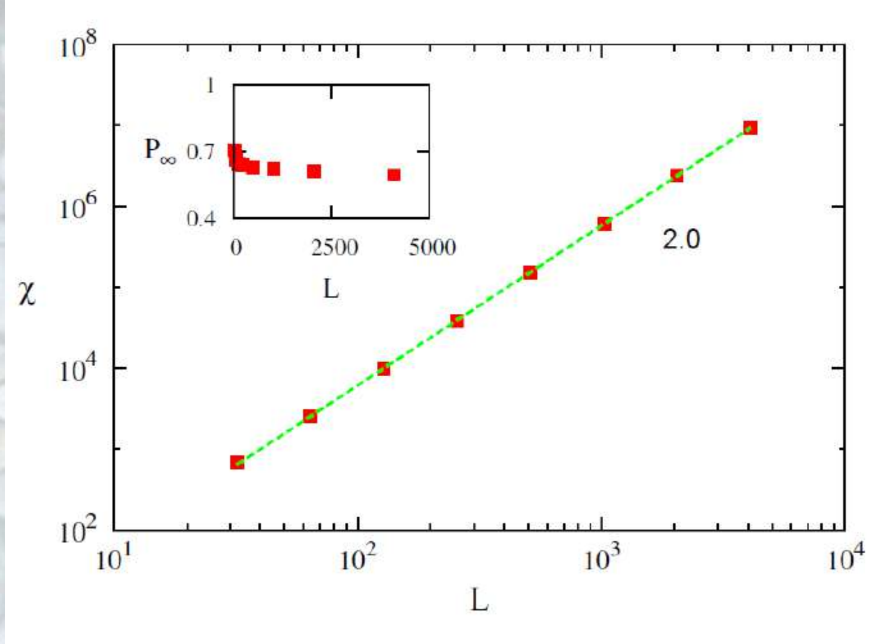
Best-of-*m* Model

$$\chi = \sum_i{}' s_i^2$$

$$P_\infty = s_{\max} / N$$

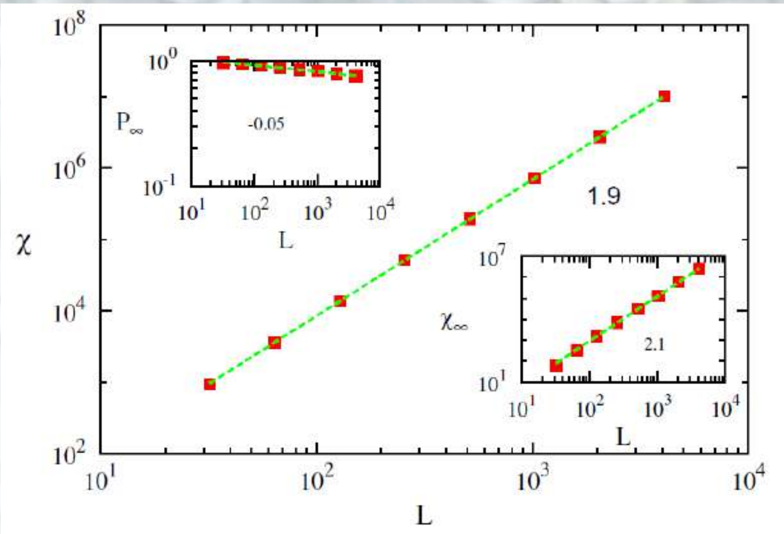
$$\chi_\infty = \sqrt{\langle s_{\max}^2 \rangle - \langle s_{\max} \rangle^2}$$

$m = 10$

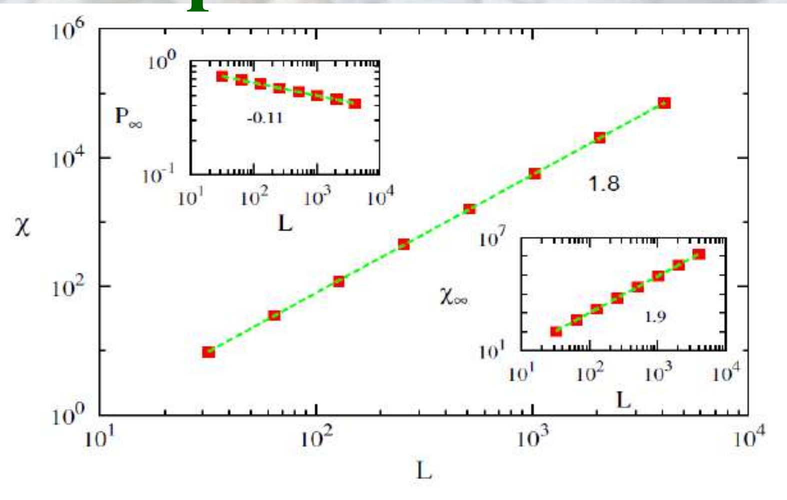


at p_c on square lattice

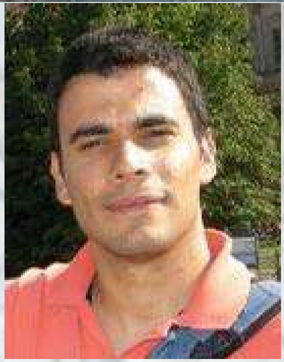
$m = 2$



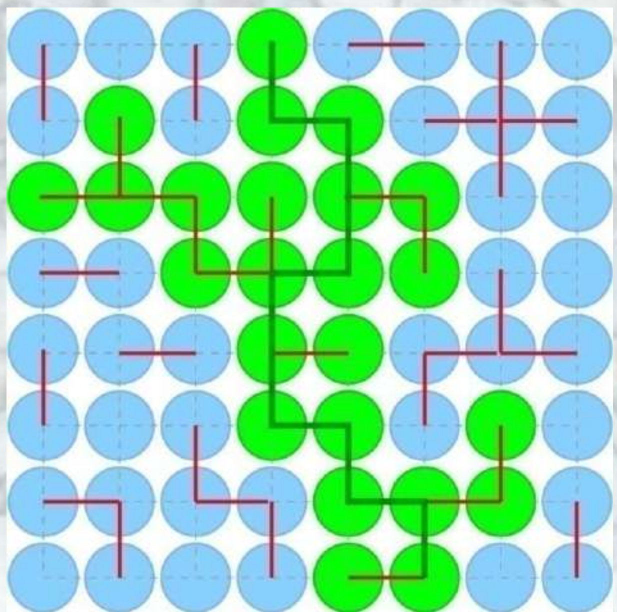
classical percolation



Largest Cluster Model



Nuno Araújo



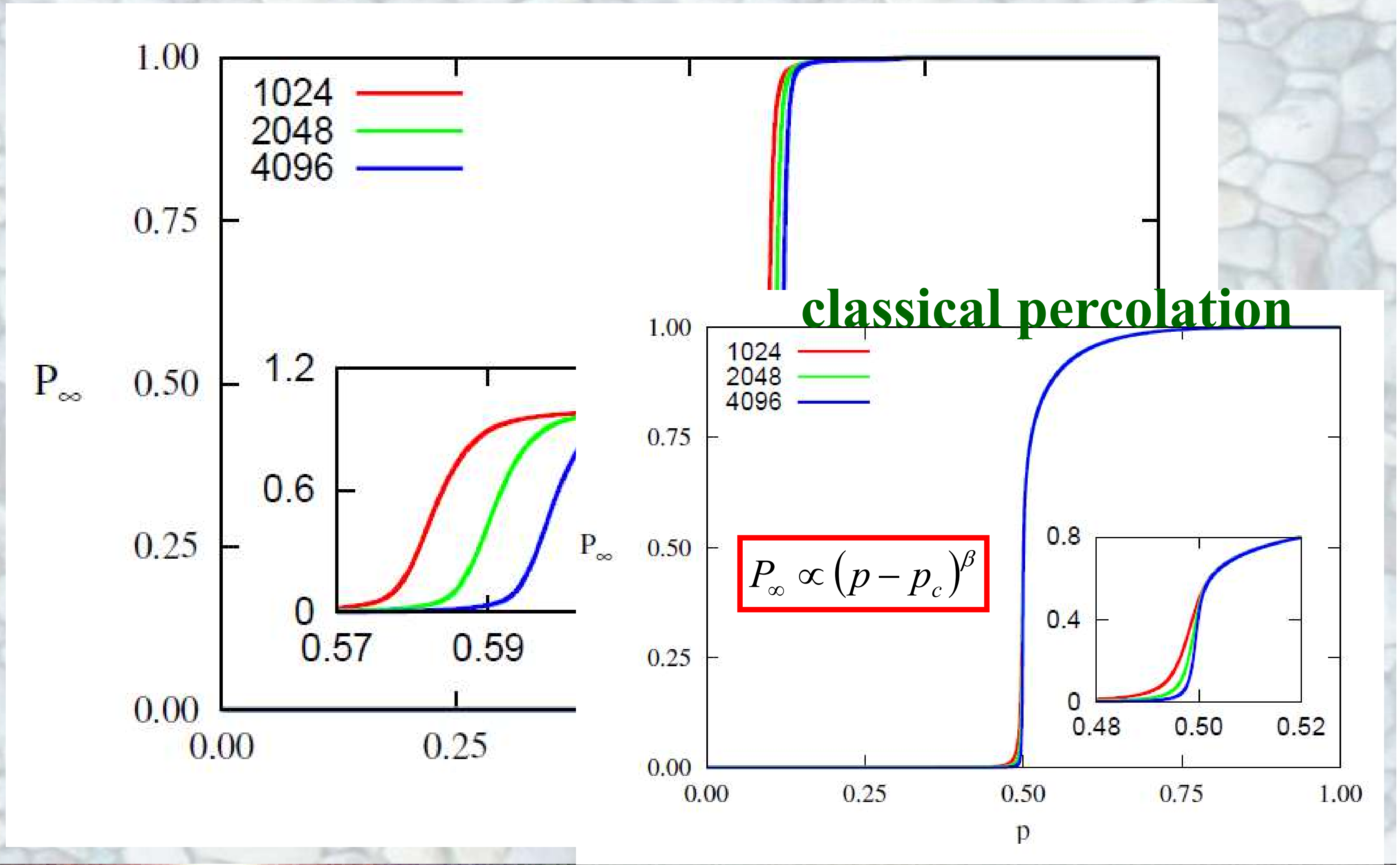
- **select randomly a bond**
- **if not related with the largest cluster occupy it**
- **else, occupy it with probability**

$$q = \exp\left[-\left(\frac{s-\bar{s}}{\bar{s}}\right)^2\right]$$

Nuno Araújo and HJH, *Phys. Rev. Lett.* 105, 035701 (2010)

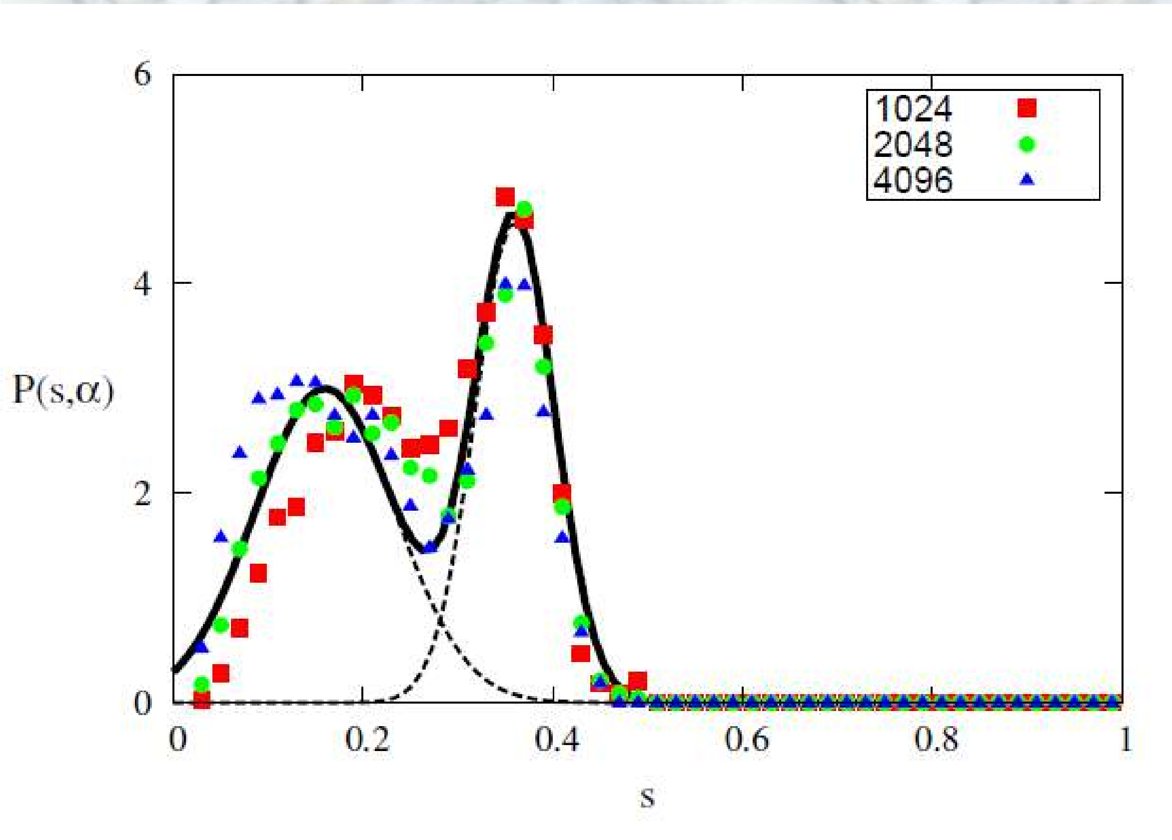
Largest Cluster Model

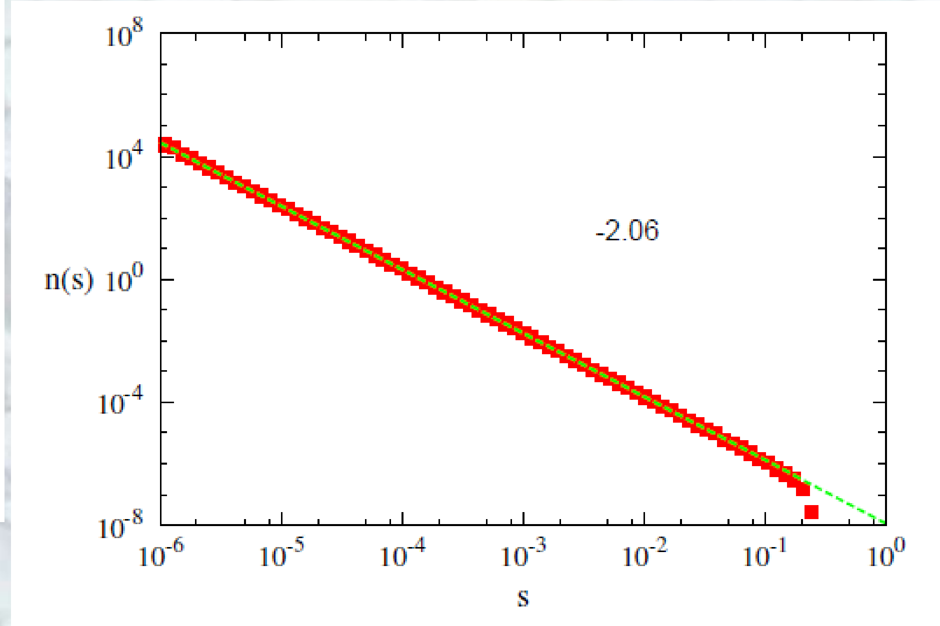
order parameter: P_∞ = fraction of sites in largest cluster



Largest Cluster Model

at p_c cluster size distribution

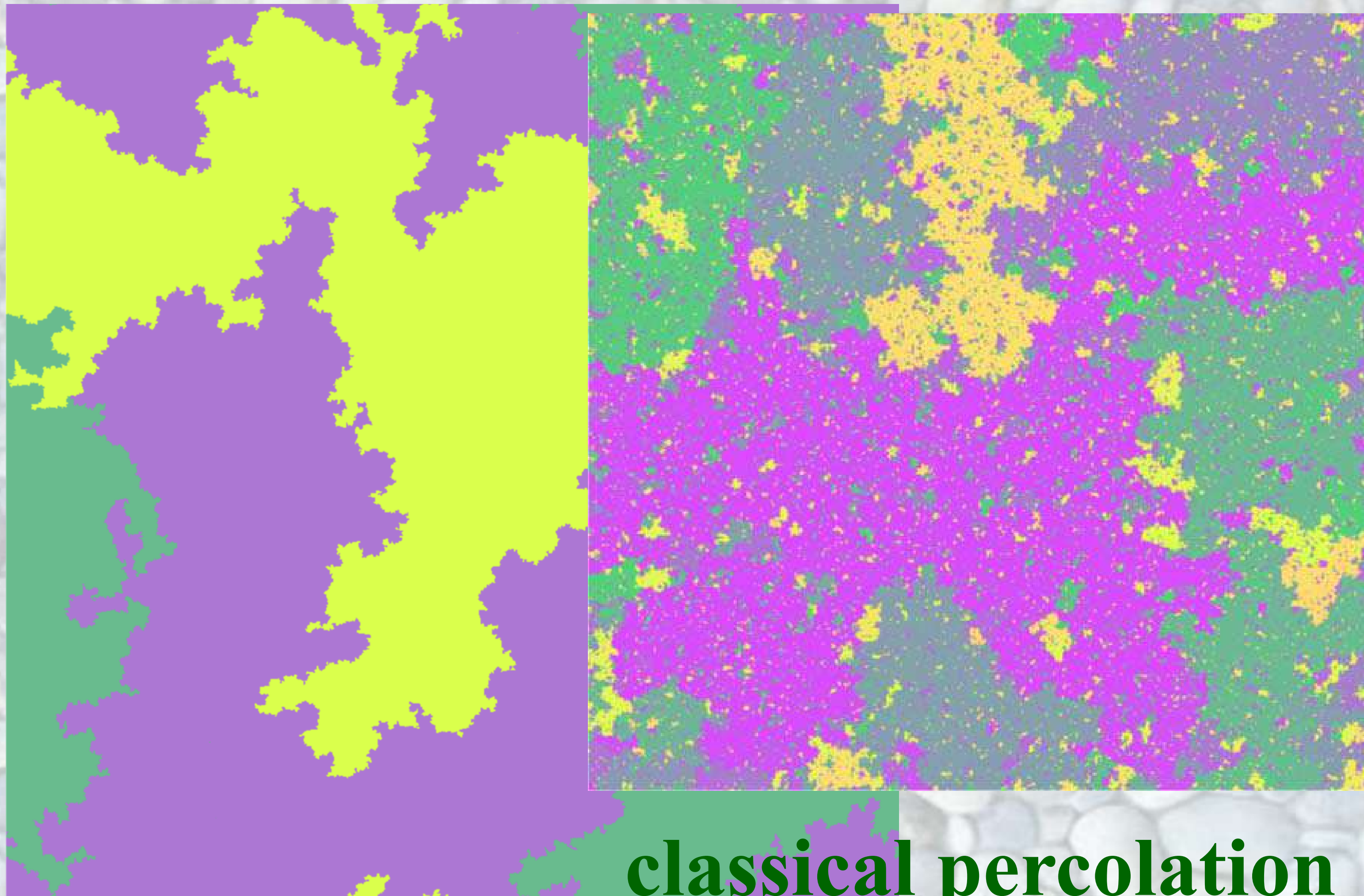




classical percolation

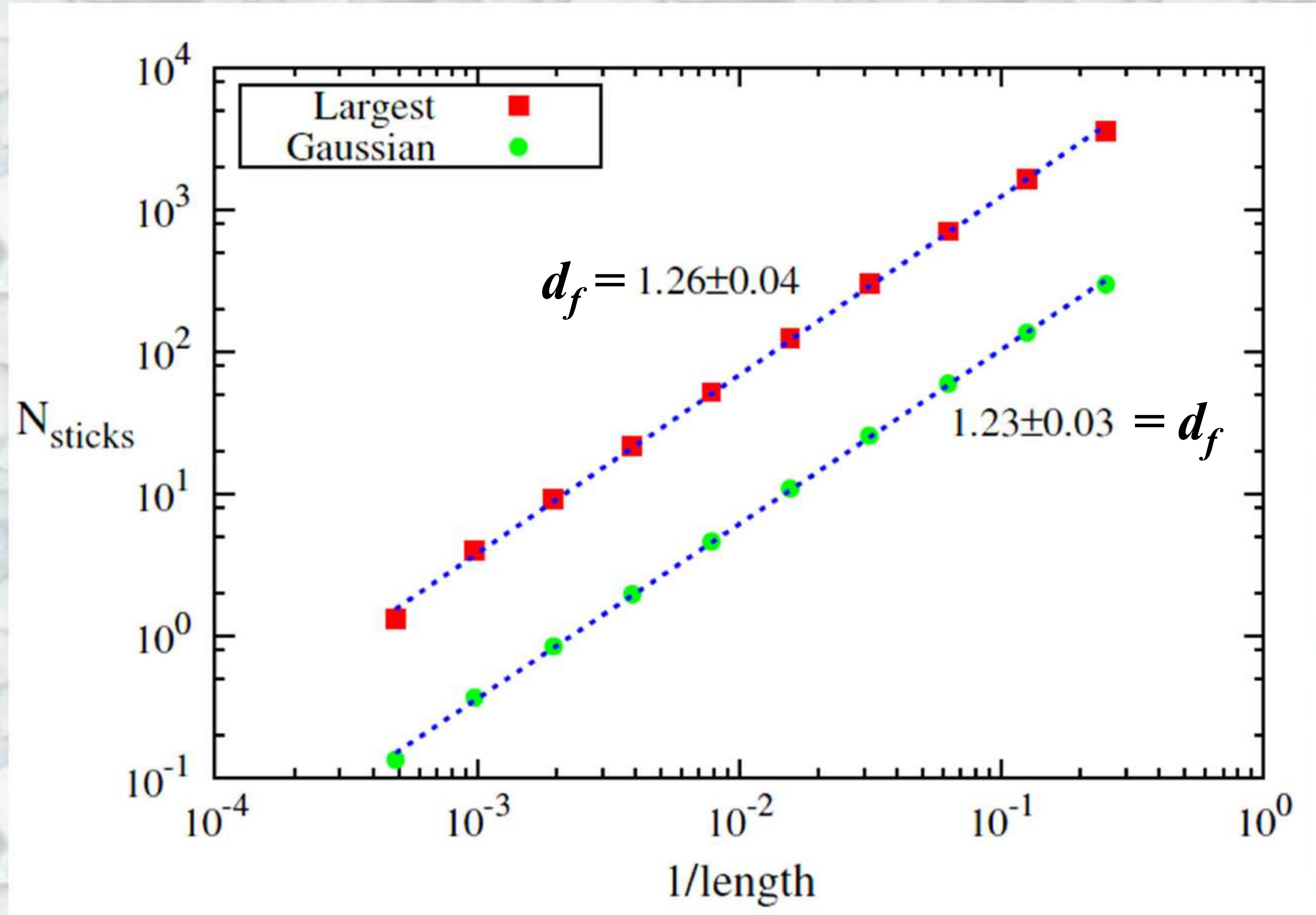
Largest Cluster Model

at p_c



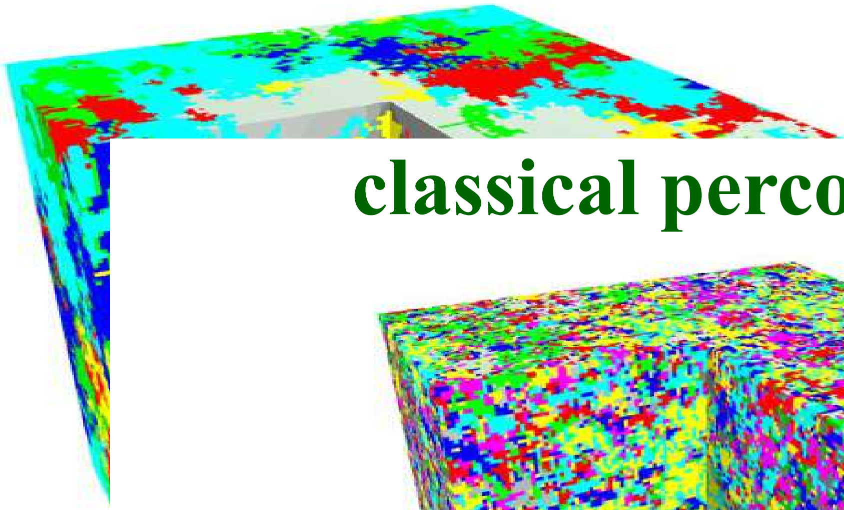
classical percolation

Surface of the clusters

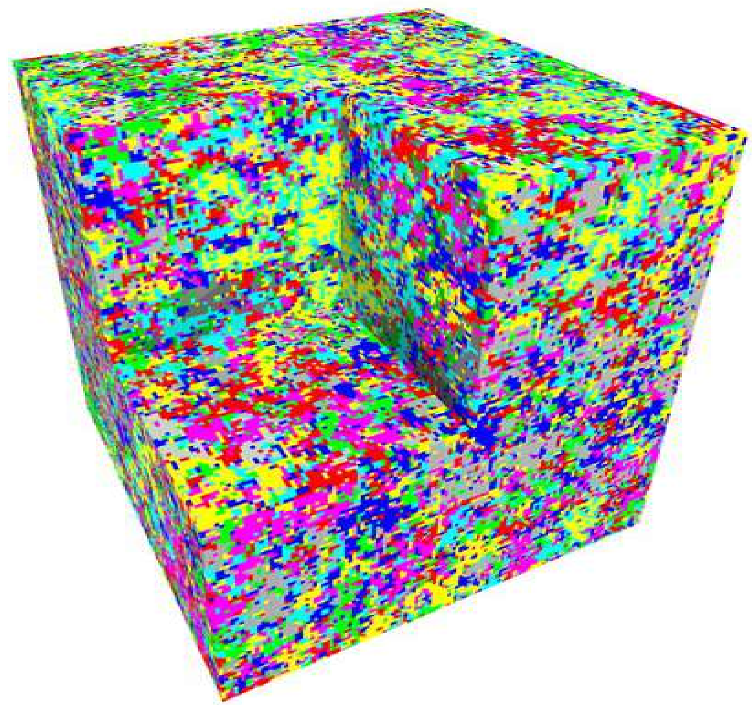


Largest cluster Model in 3D

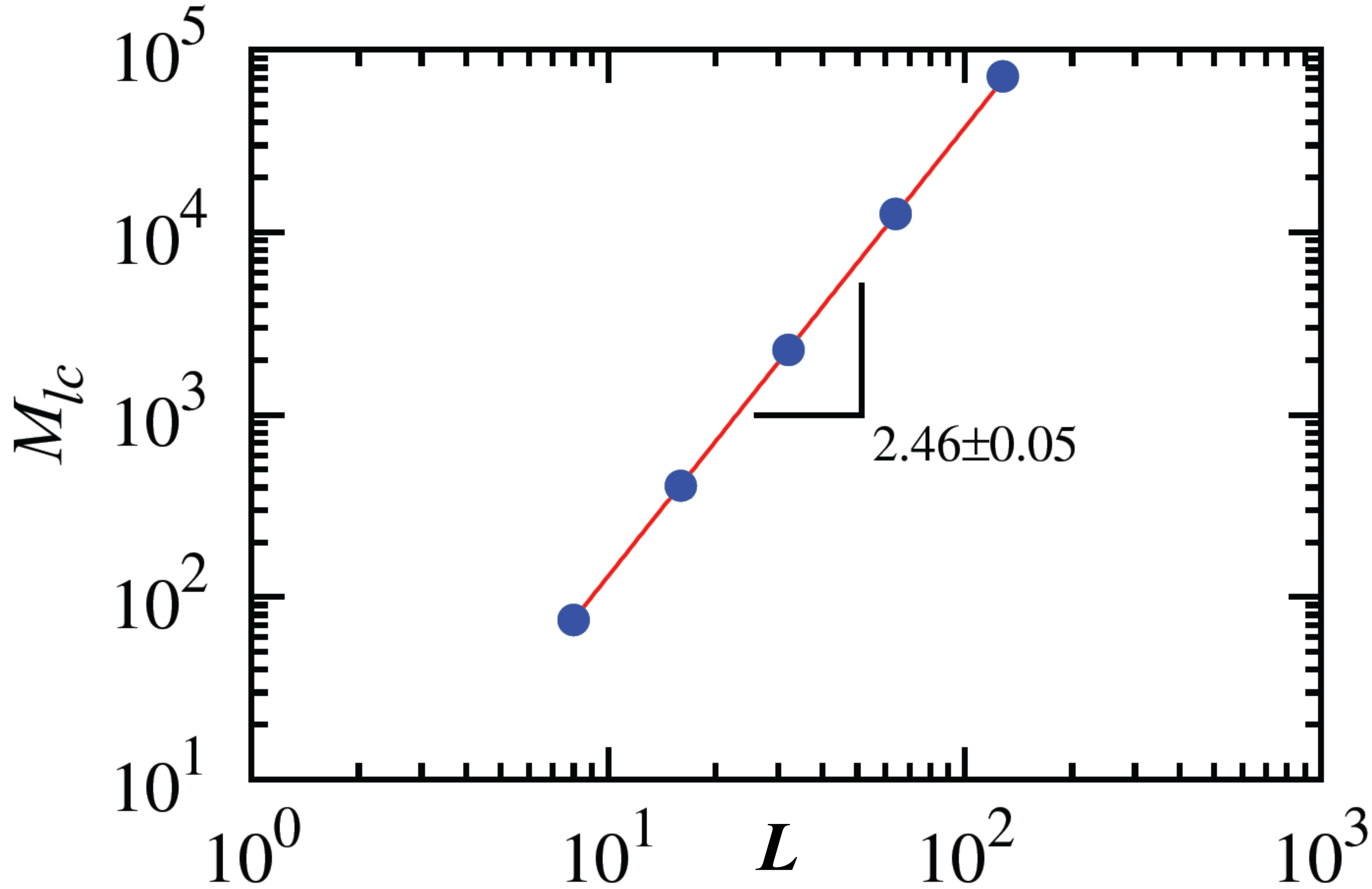
K.J. Schrenk, N.A.M. Araújo, and H.J.H.,
Phys. Rev. E, 84, 041136 (2011)



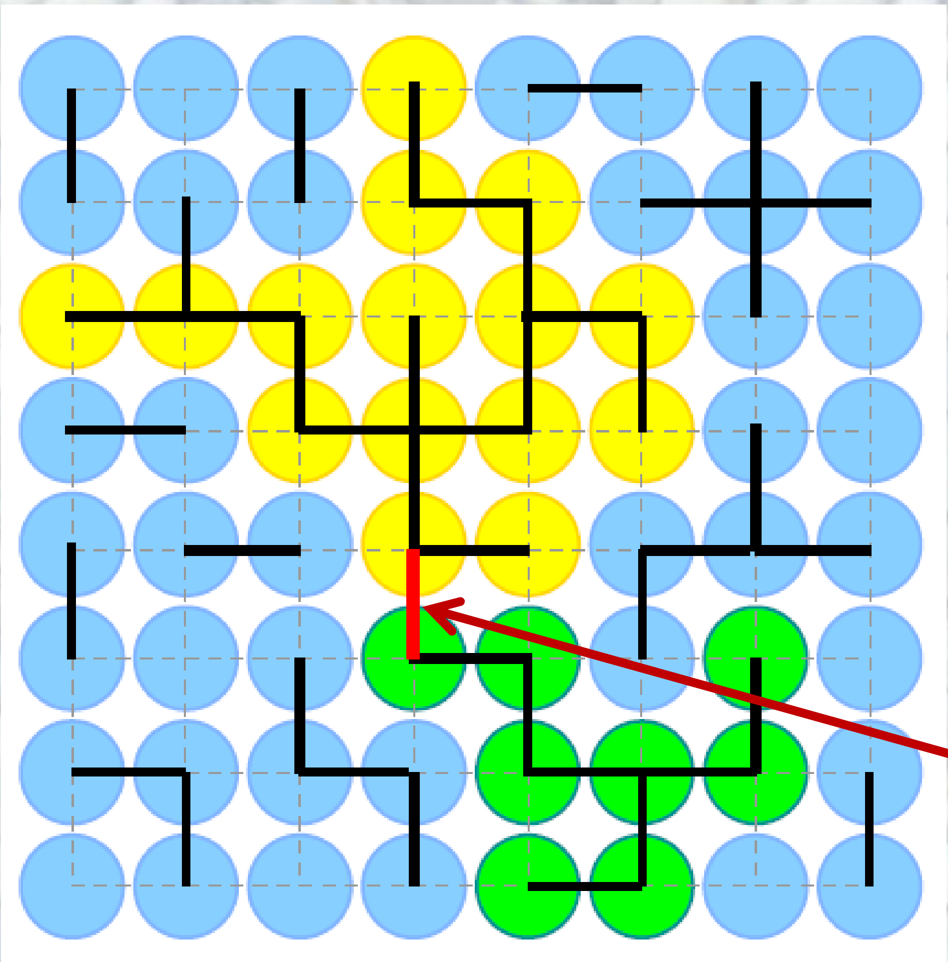
classical percolation



Largest cluster model in 3D



Bridge Percolation

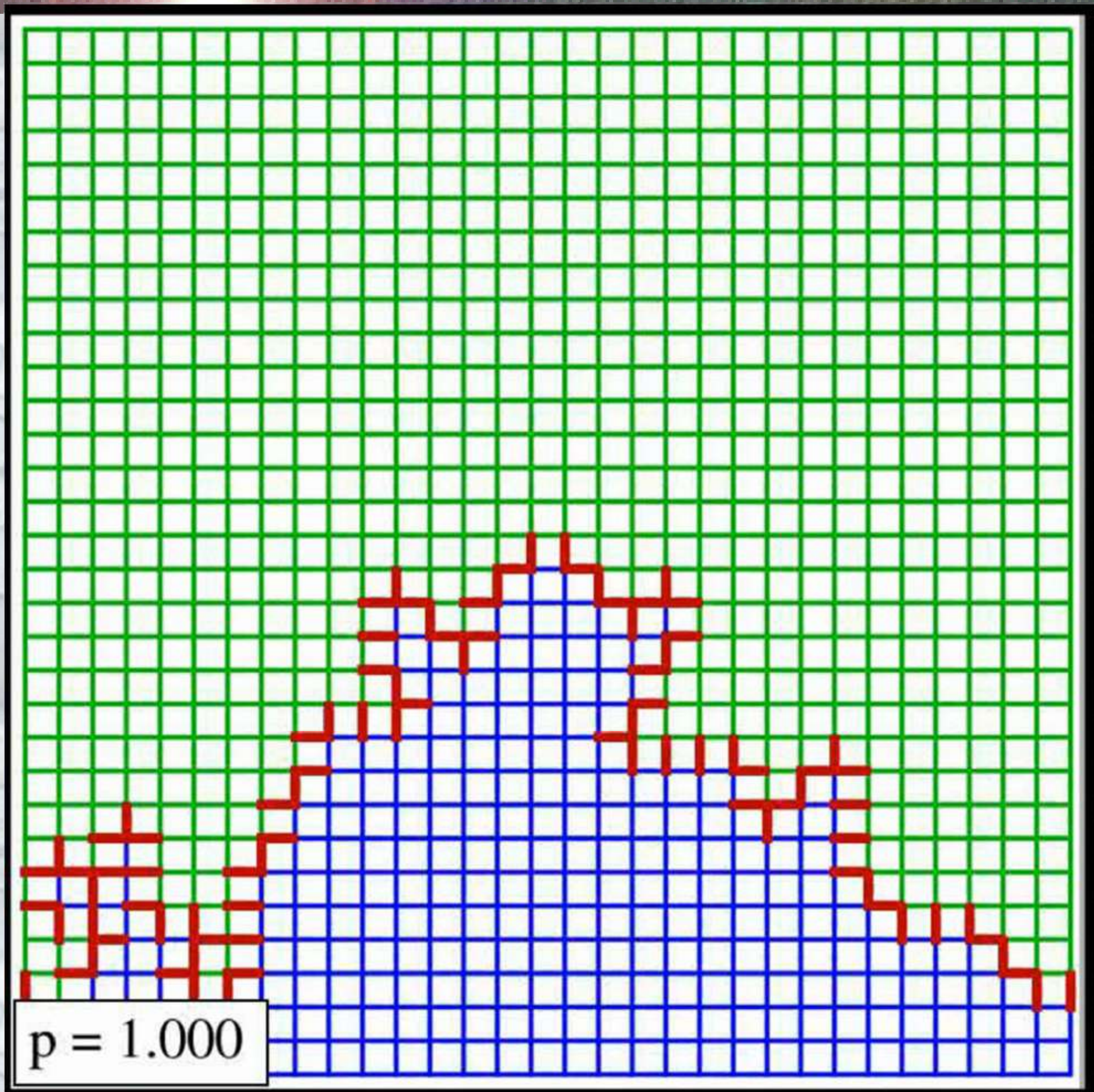


A bridge (or anti-red bond) is a bond which if occupied would create the first spanning cluster.

bridge

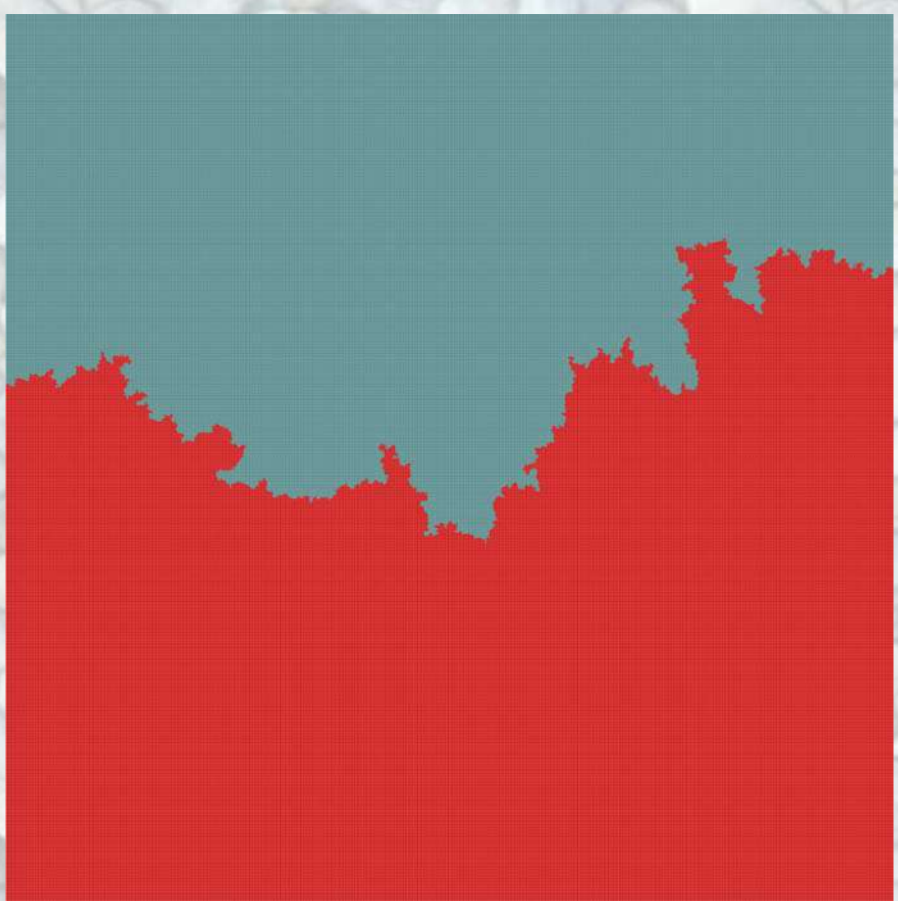
K.J. Schrenk, N.A.M. Araújo, J.S. Andrade Jr., H.J.H., Sci. Rep. 2, 348 (2012)

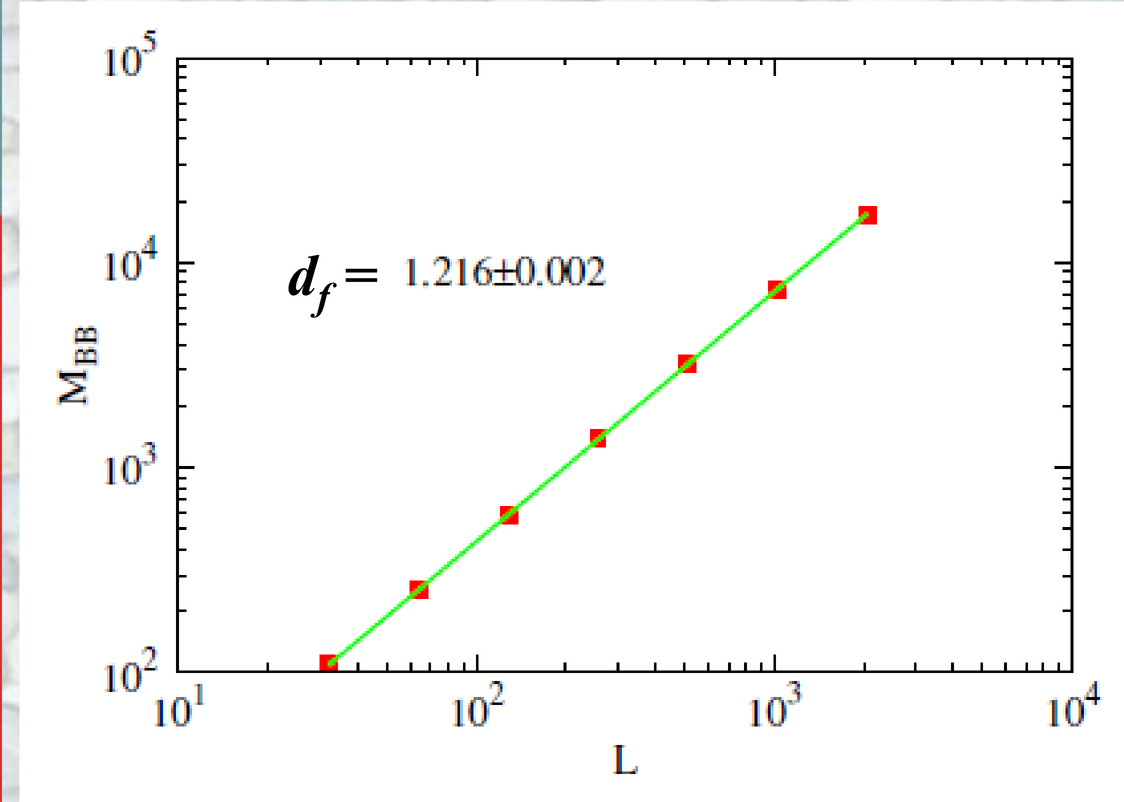
Bridge Percolation



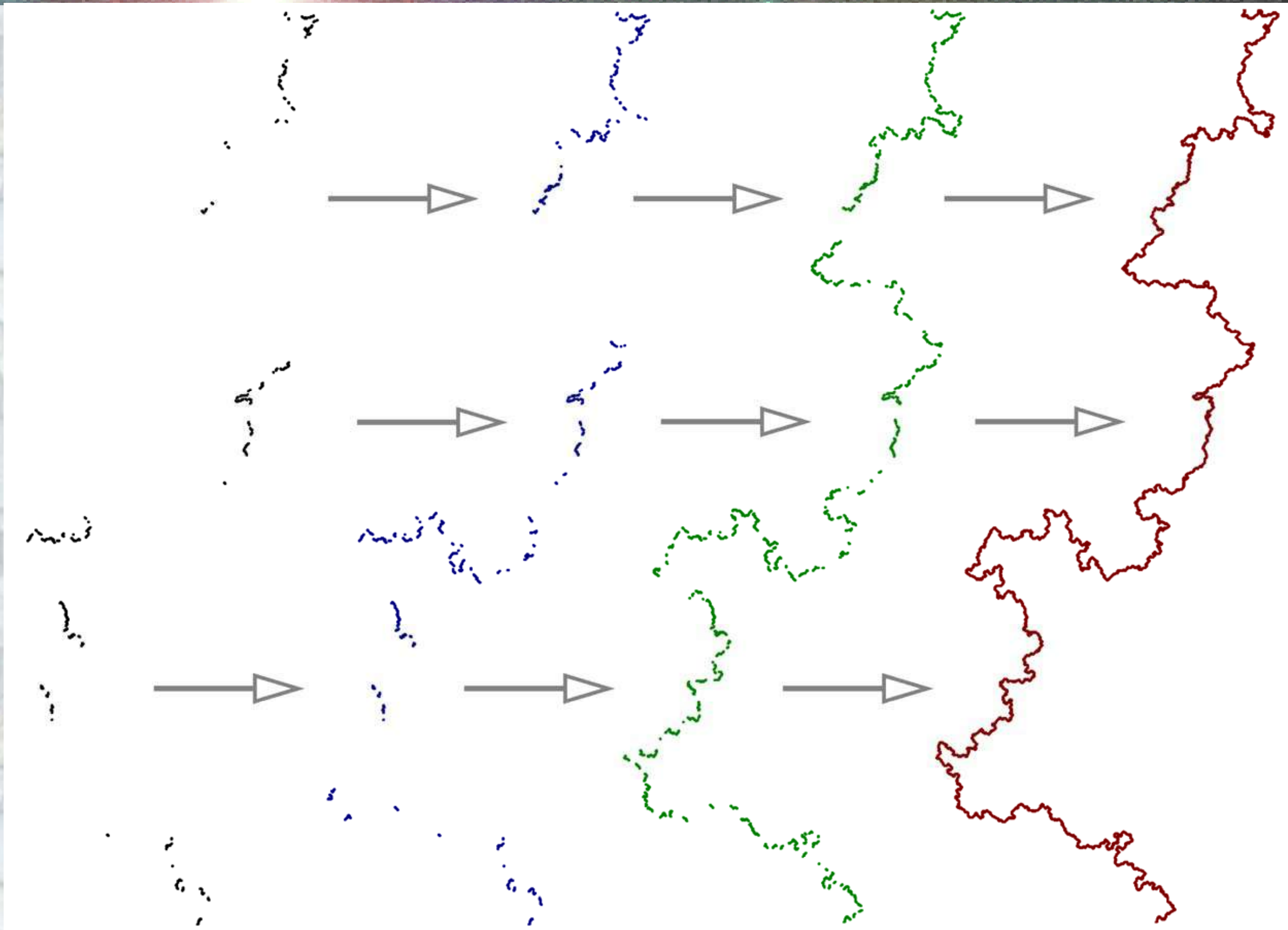
Bridge Percolation

$p_{\text{c-bridge}} = 1$



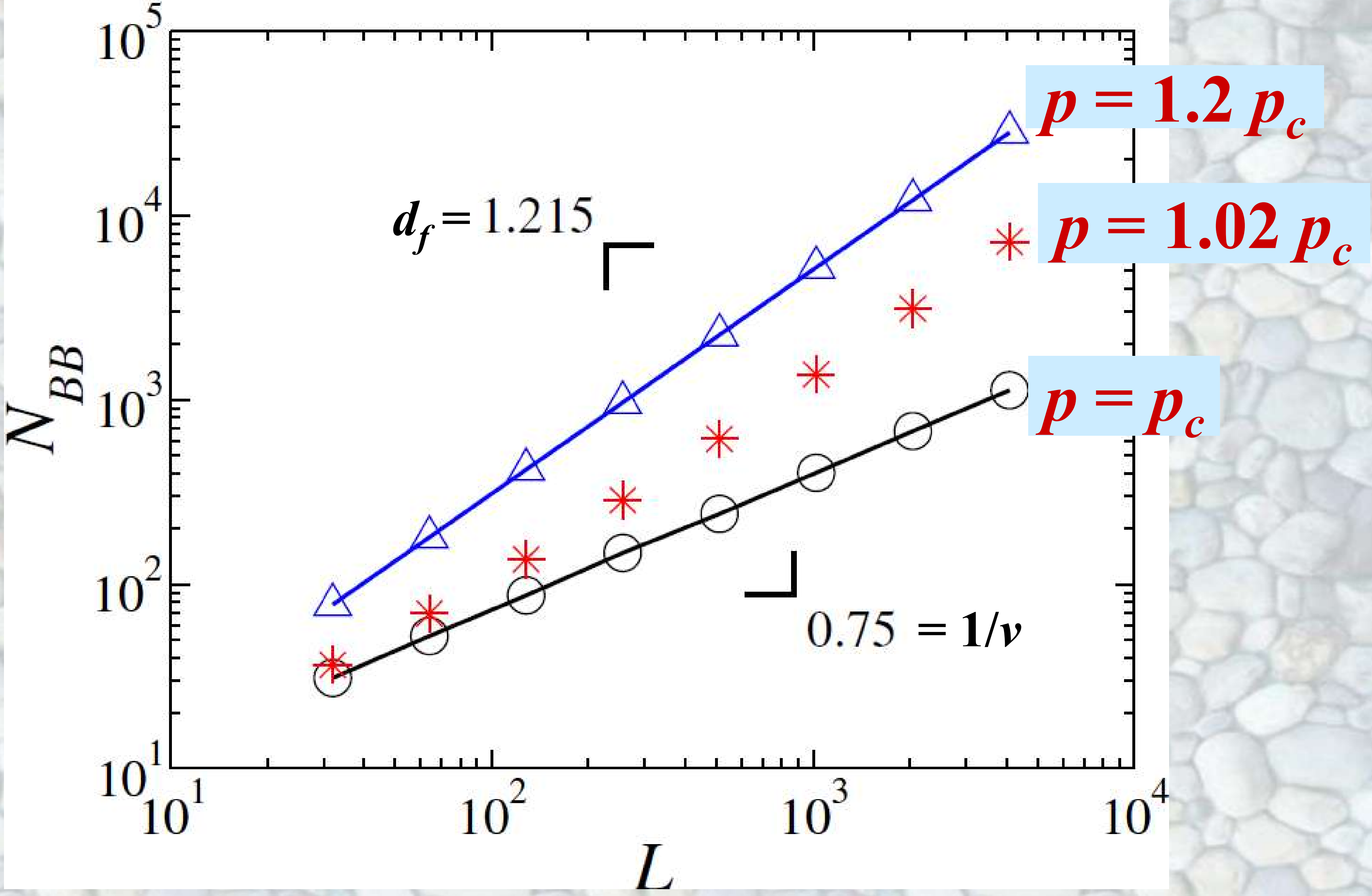


Bridge Percolation



$p = p_c$ | $p = 1.01\, p_c$ | $p = 1.05\, p_c$ | $p = 1$

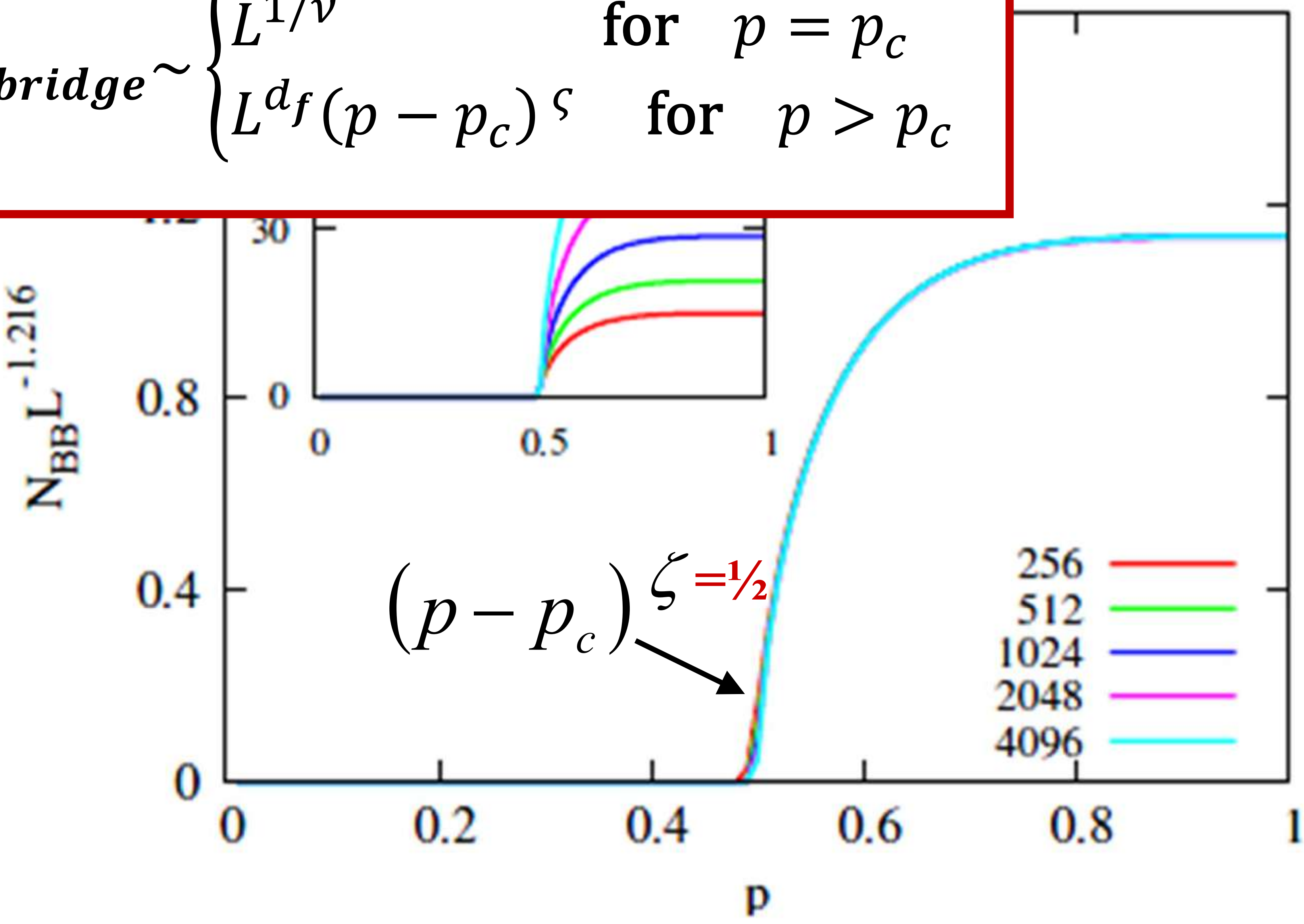
Bridge Percolation



Bridge Percolation

$$N_{bridge} \sim \begin{cases} L^{1/\nu} & \text{for } p = p_c \\ L^{d_f}(p-p_c)^{\varsigma} & \text{for } p > p_c \end{cases}$$

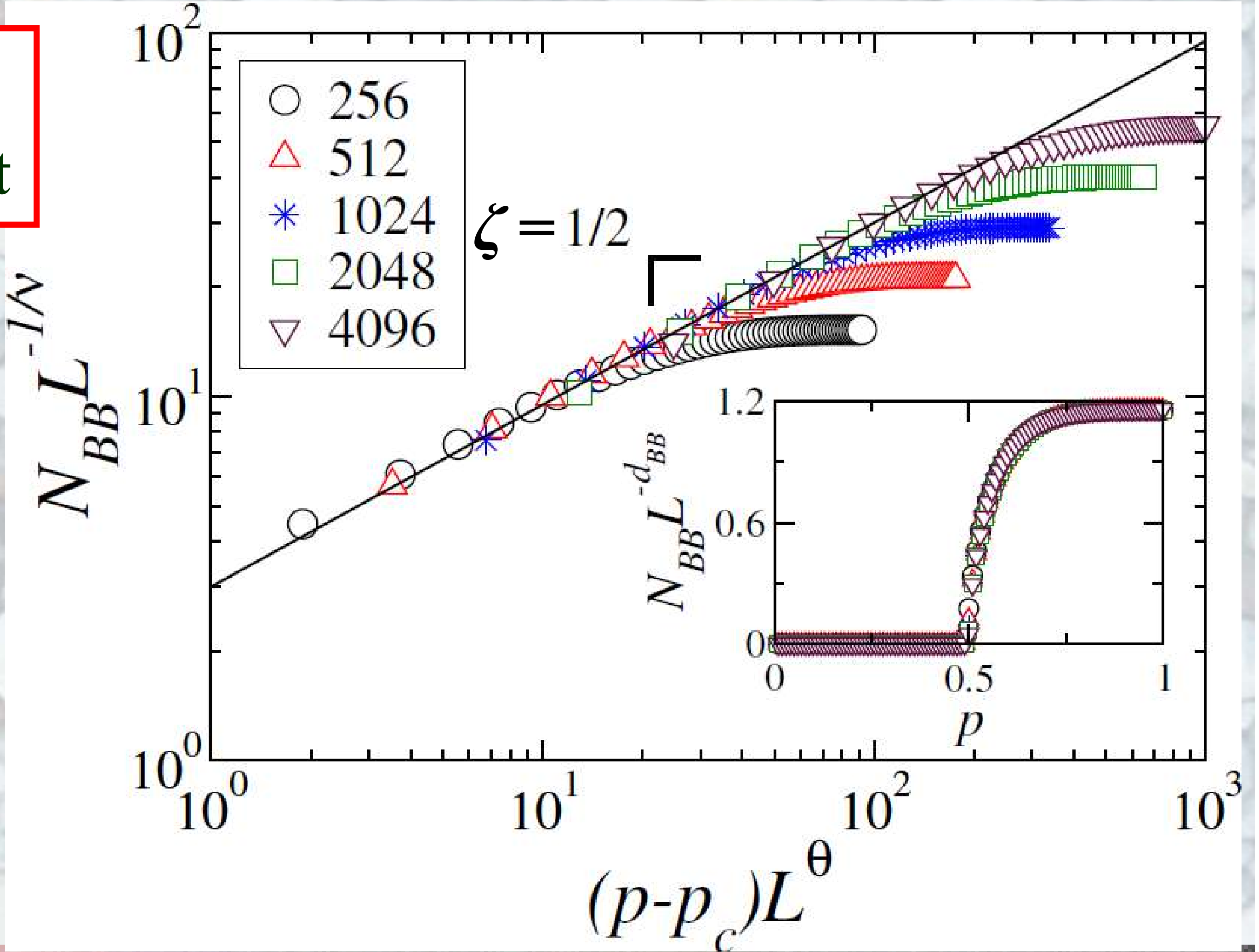
N_{bridge} **is number of bridge bonds.**



Bridge Percolation

$$N_{bridge}(p,L) = L^{\frac{1}{\nu}} \mathrm{F}\left[(p-p_c)L^{\theta}\right] = L^{d_f} \tilde{\mathrm{F}}\left[(p-p_c)L^{\varsigma}\right]$$

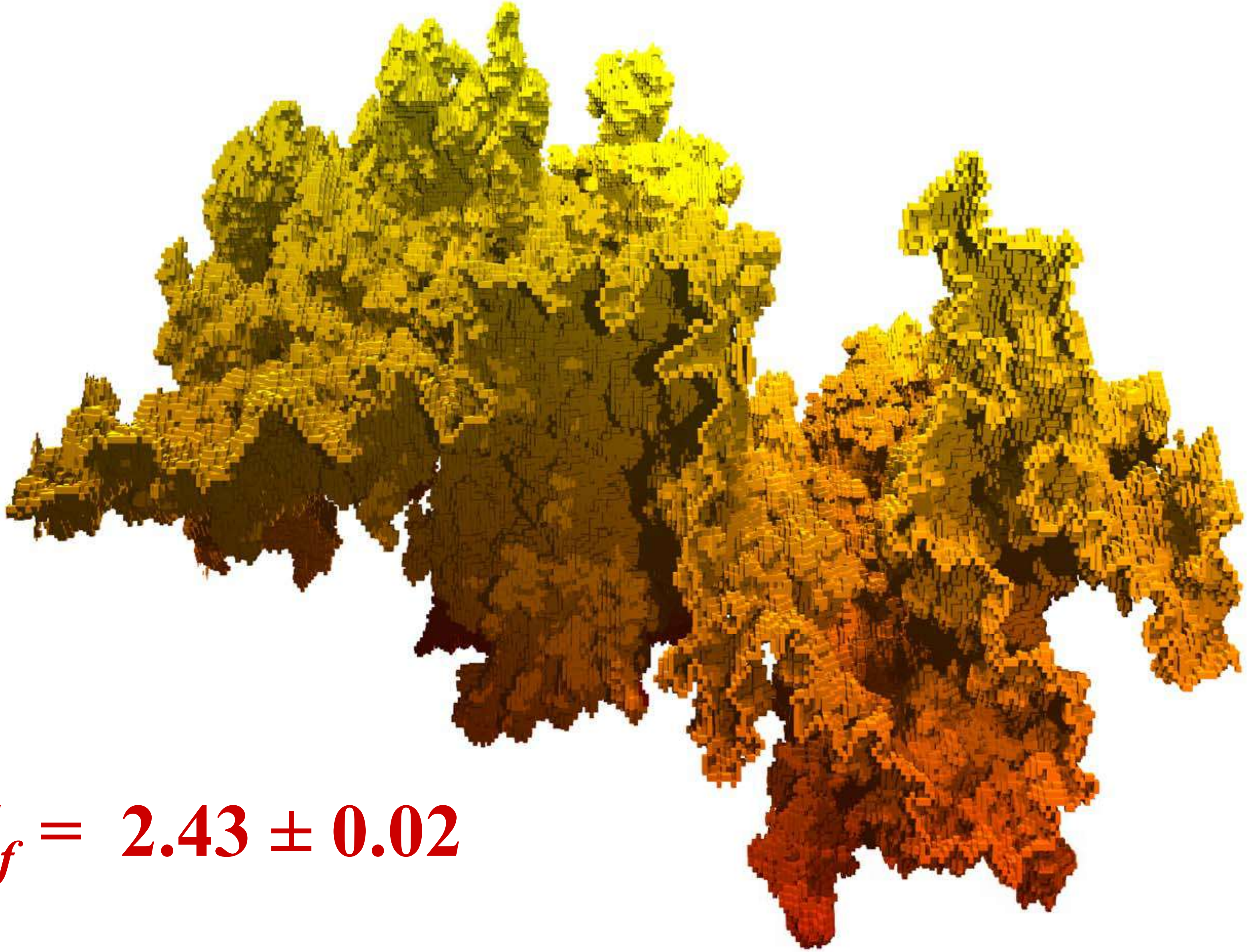
Tricritical scaling analogous to theta point



$$\theta = \zeta^{-1}\left(d_f - \frac{1}{\nu}\right)$$

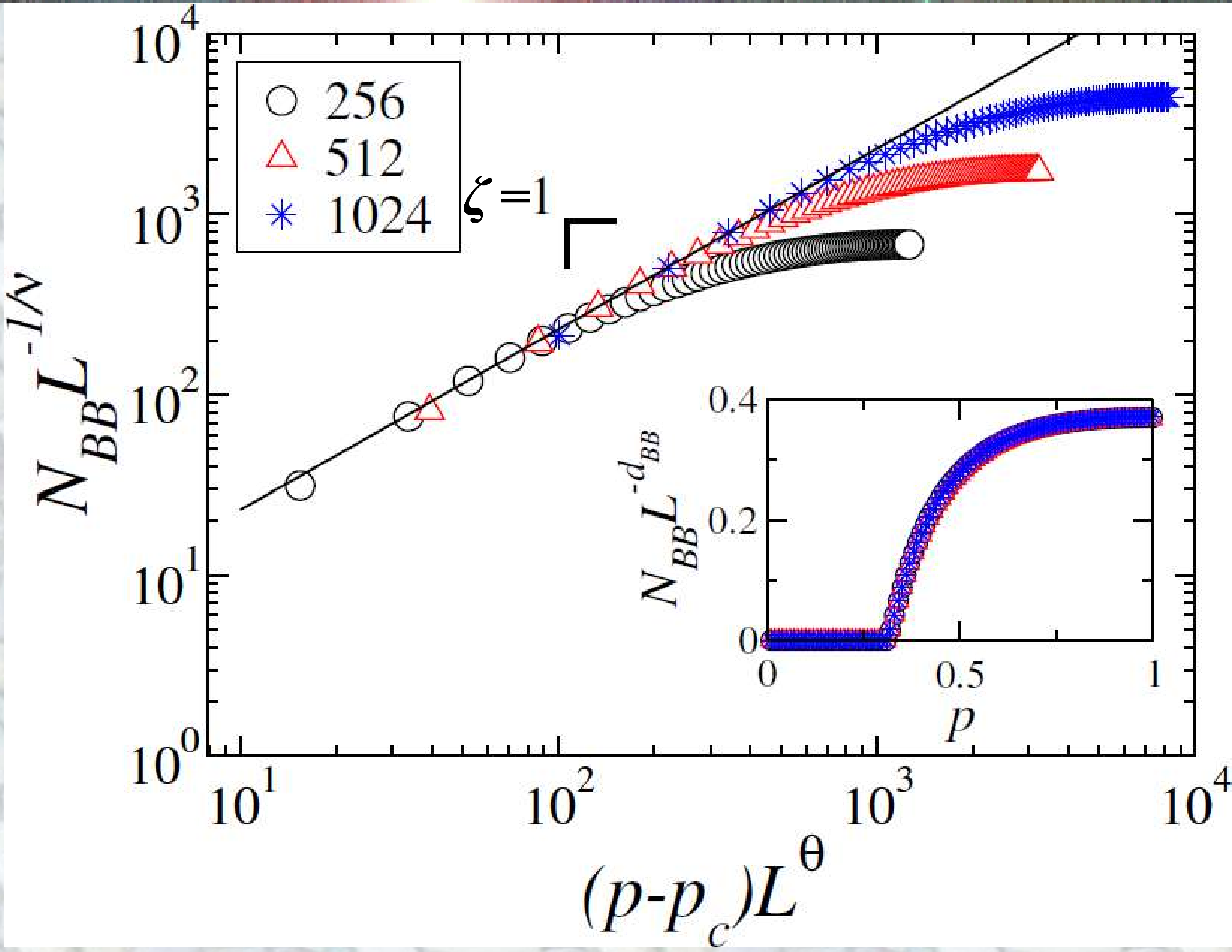
θ = 0.94

Bridge Percolation in 3D



$d_f = 2.43 \pm 0.02$

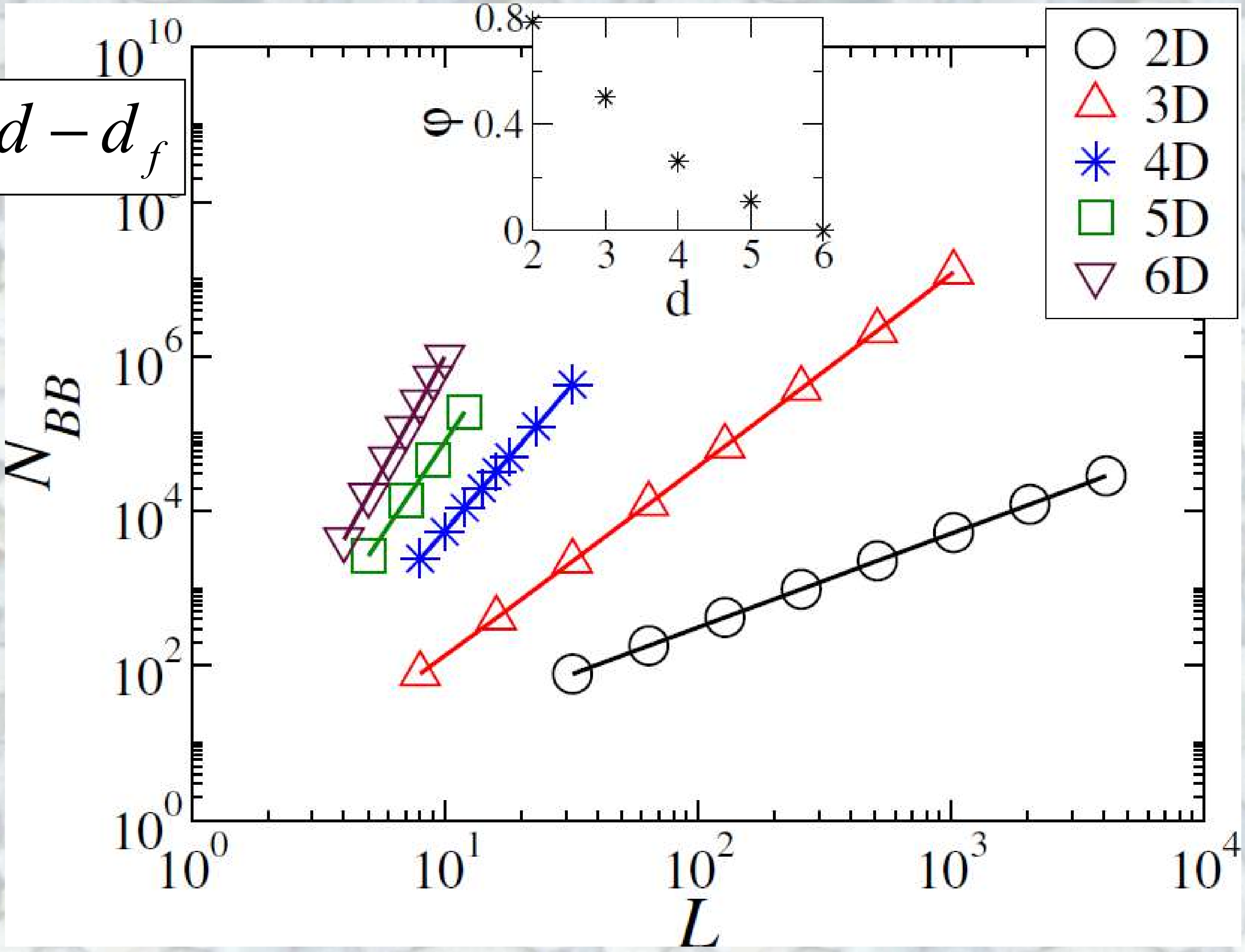
Bridge Percolation in 3D



θ = 1.36

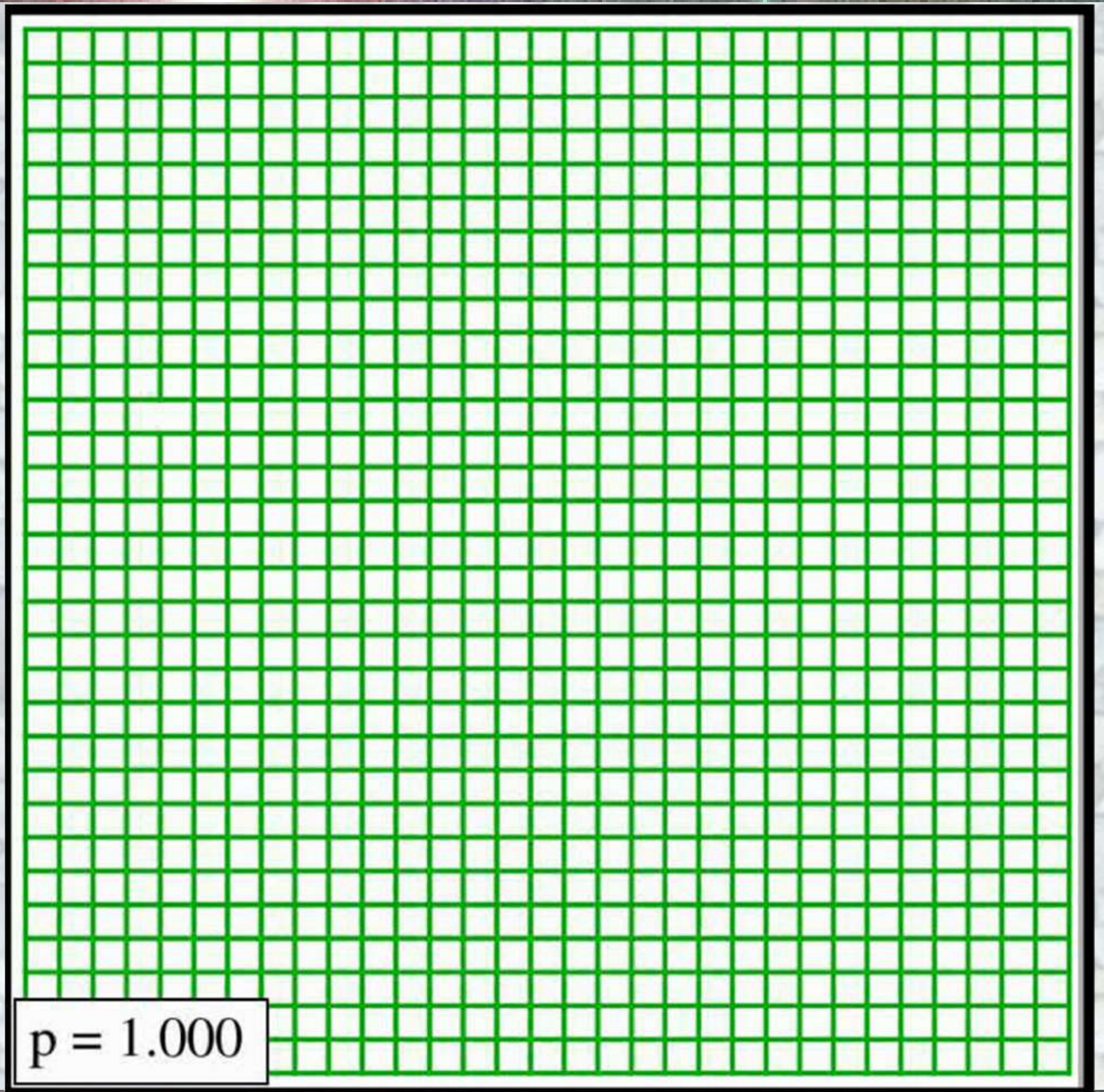
Bridge Percolation $d = 2 - 6$

$$\varphi = d - d_f$$



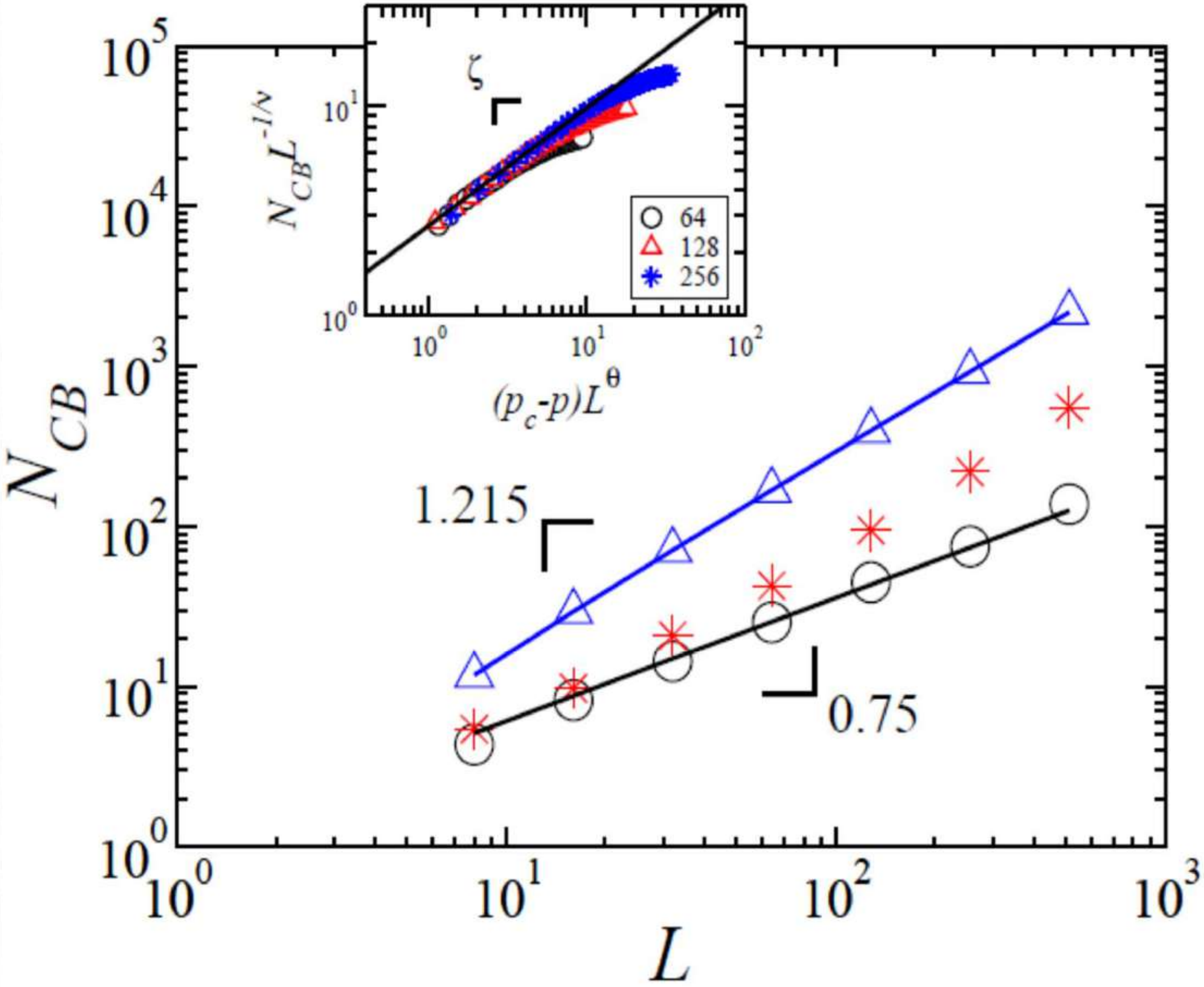
Above the upper critical dimension $d_c = 6$ the set of bridges is dense.

Cutting bonds



Cutting bonds

If one starts from a fully occupied lattice and removes bonds except if they are cutting bonds in 2d they have the same behavior as the bridges before (same exponents). In higher dimension the exponents are different.

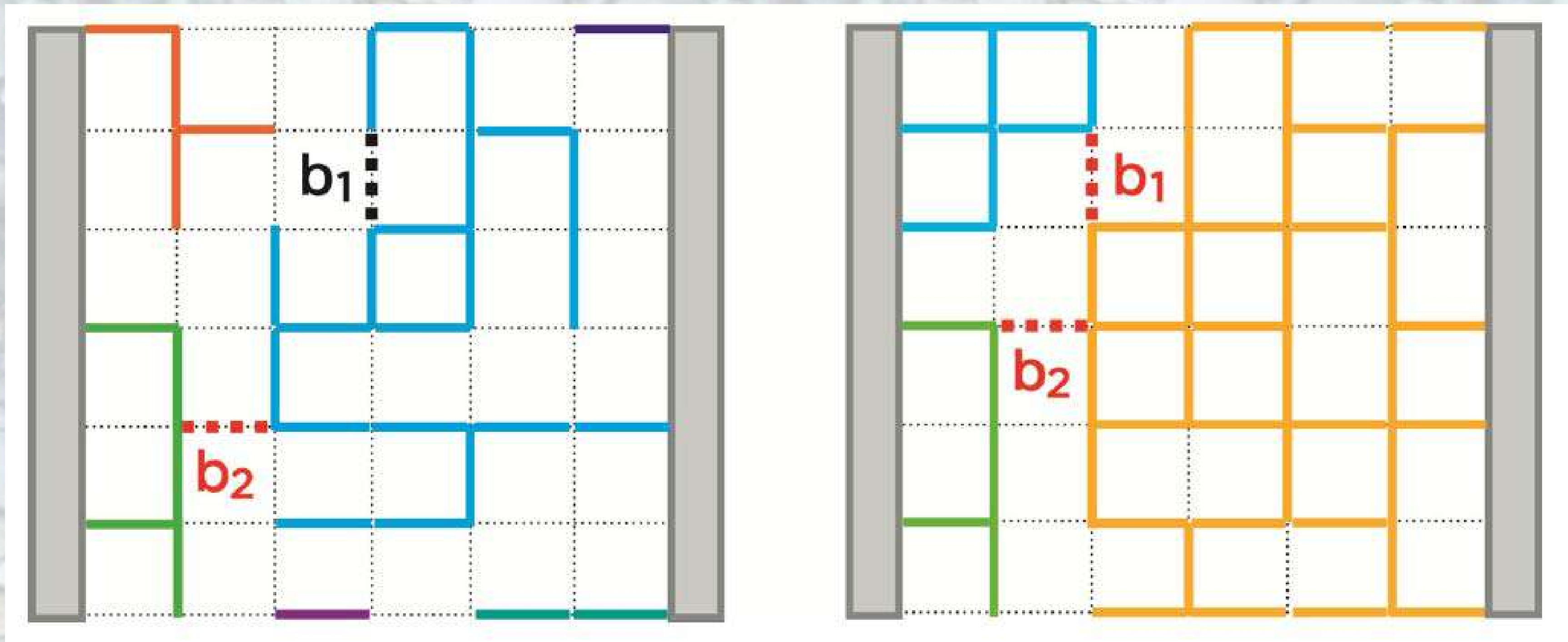


Spanning cluster avoiding model

Y. S. Cho, S. Hwang, H.J.H., and B. Kahng, Science, 339, 1185 (2013)

Choose *m* unoccupied bonds and occupy randomly one which is not a bridge, if all are bridges then choose randomly one of these bridges.

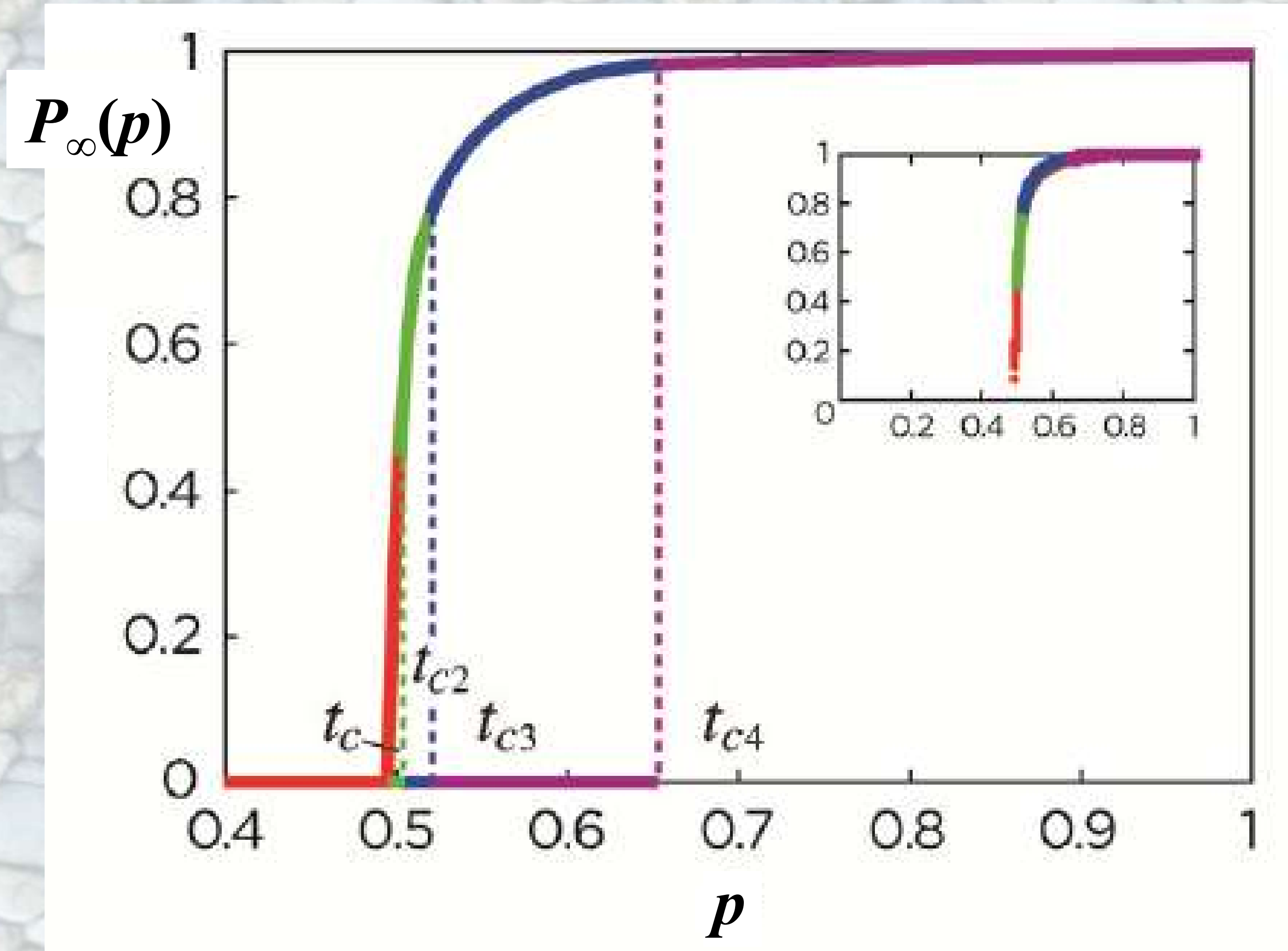
$m = 2$



Spanning cluster avoiding model

For finite systems there is a jump for *m* > 1.

fraction of bonds in the spanning cluster



$L = 1000$

Y. S. Cho, S. Hwang, H.J.H., and B. Kahng, Science, 339, 1185 (2013)

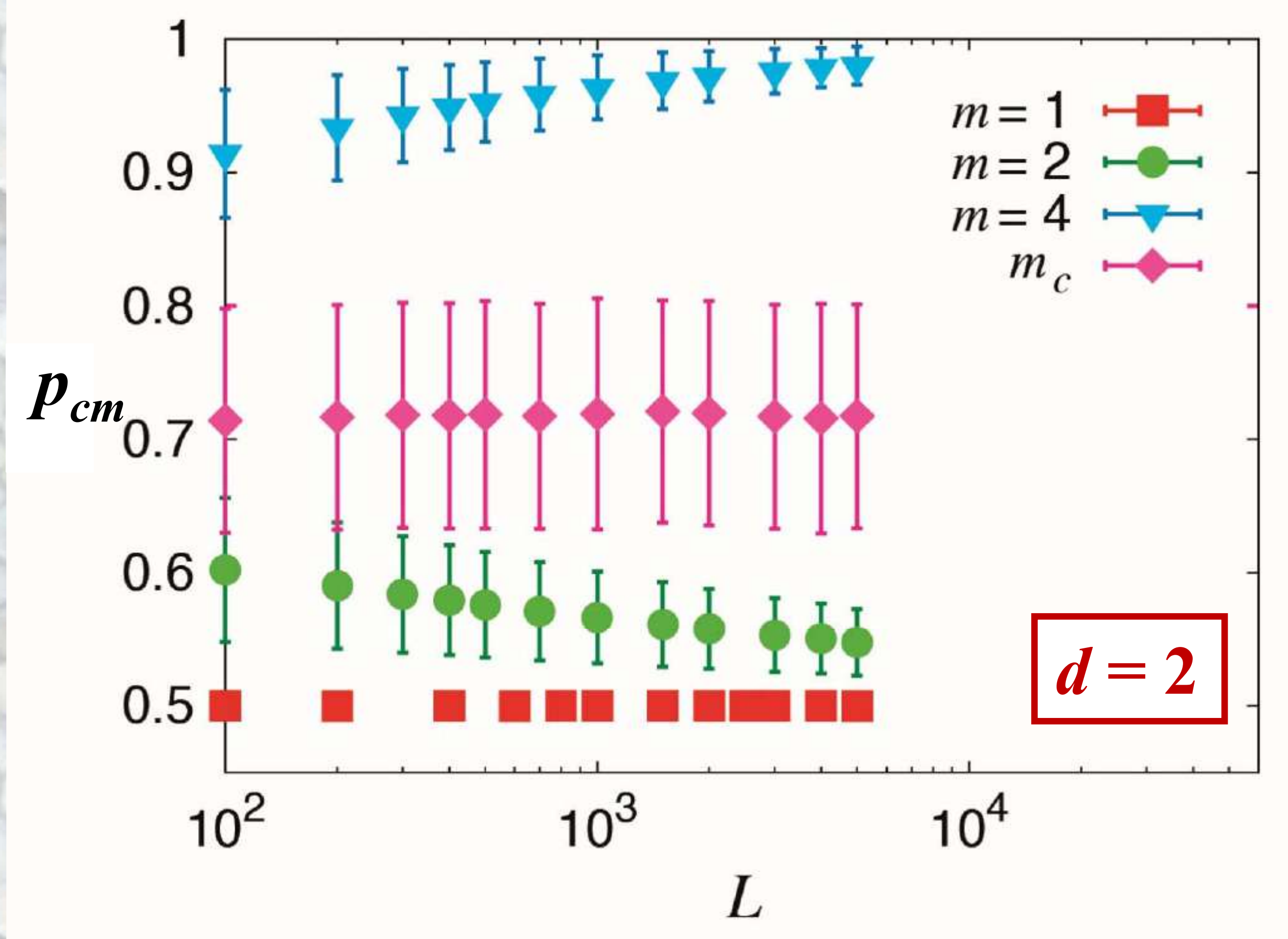
Spanning cluster avoiding model

At each dimension d there exists an m_c so that for increasing system size L the transition goes to

$p_c = p_{c\,m=1}$ for $m < m_c$

and to

$p_c = 1$ for $m > m_c$.



$m_c(2) \approx 2.55 \pm 0.01$ $\quad m_c(3) = 5.98 \pm 0.07$ $\quad m_c(4) = 16.99 \pm 5.23$

Y. S. Cho, S. Hwang, H.J.H., and B. Kahng, Science, 339, 1185 (2013)

Spanning cluster avoiding model

$N_b = d\,L^d$ is the number of bonds

$$N_{BB} \sim \begin{cases} L^{1/\nu} & \textbf{for} \quad p = p_c \\ L^{d_f}(p-p_c)^{\varsigma} & \textbf{for} \quad p > p_c \end{cases}$$

probability to have *m* bridge bonds:

$$q(p,m) = \left[\frac{N_{BB}}{N_b(1-p)}\right]^m \sim N_b^{-m}\left[\frac{\boldsymbol{N_b}^{\boldsymbol{d_f}/\boldsymbol{d}}(p-p_c)^{\varsigma}}{1-p}\right]^m$$

$$\Rightarrow \quad m_c(d) = \frac{d}{d-d_f} \quad \Rightarrow$$

For *d* > 6 the transition is always continuous.

Spanning cluster avoiding model

One can also show analytically that:

for $m < m_c$

$$p_{cm}(N) - p_c \sim N^{-1/\bar{\nu}_<}$$

$$1/\bar{\nu}_< = (1 - m/m_c)/(m\zeta + 1),$$

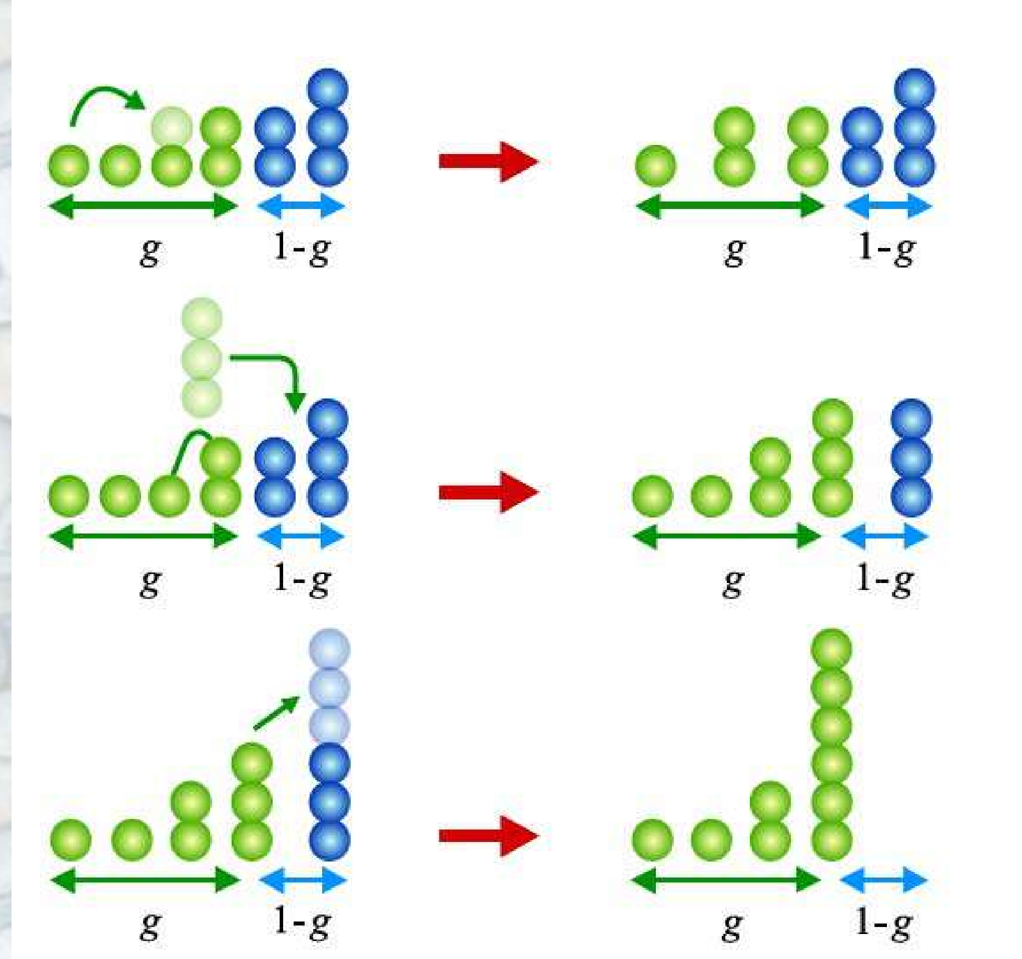
for $m > m_c$

$$1 - p_{cm}(N) \sim N^{-1/\bar{\nu}_>}$$

$$1/\bar{\nu}_> = (m/m_c - 1)/(m - 1)$$

Connecting the Disconnected

Connect randomly individuals but with a law imposing that every new connection must at least involve one individual belonging to the fraction *g* of the most disconnected population.



Y.S. Cho, J.S. Lee, H.J.H., B. Kahng, Phys. Rev. Lett. 116, 025701 (2016)

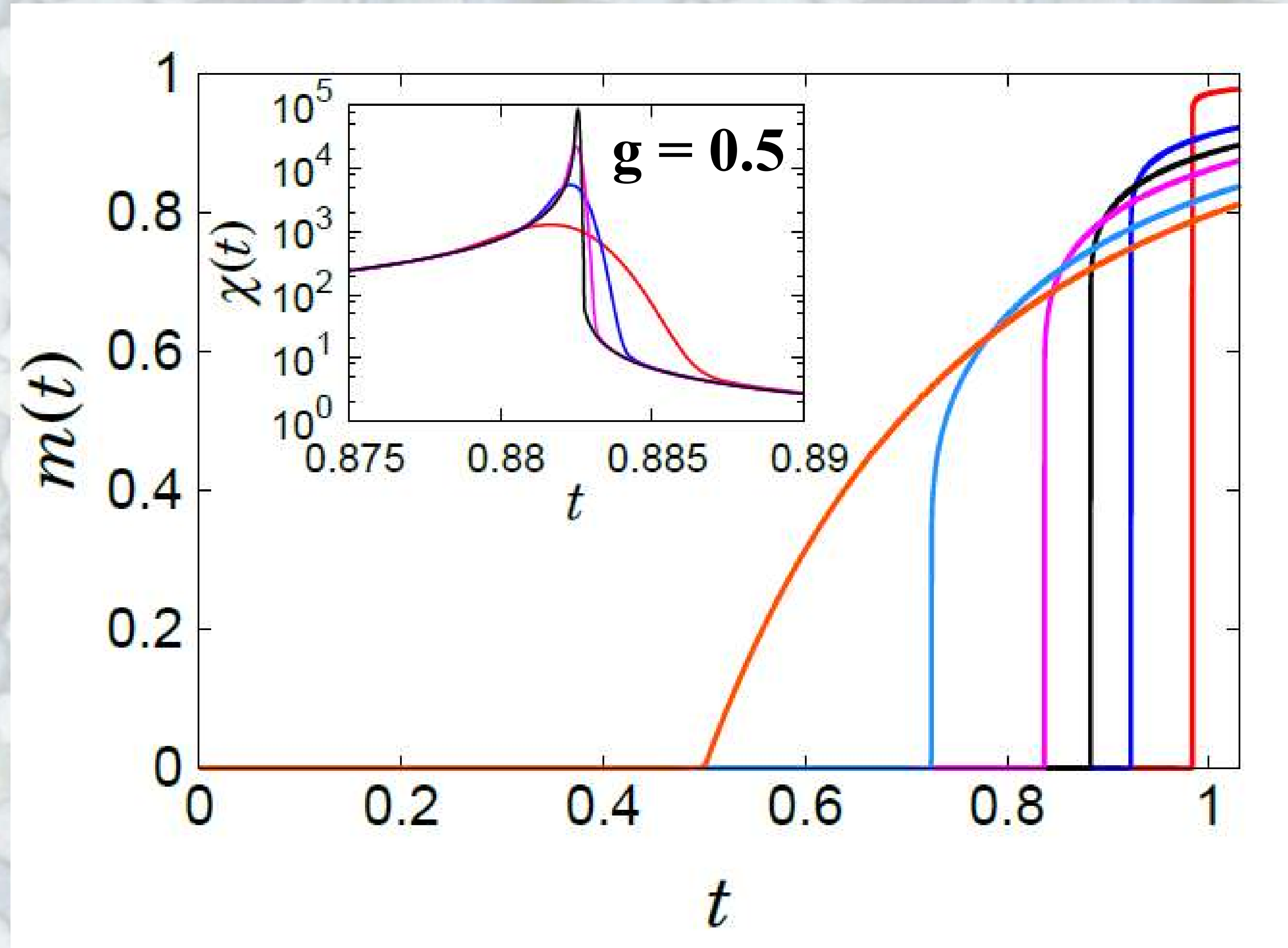
Connecting the Disconnected

- Start with *N* isolated individuals.
- R is the subset of sites belonging to the k clusters following

$$N_{k-1}(t) < [gN] \leq N_k(t) \quad \text{with} \quad N_k(t) = \sum_{l=1}^{k} s_l(t)$$

- At each step select uniformly at random one node from R and the other from the entire system.

Connecting the Disconnected



hybrid transition

***g* = 1, 0.8, 0.6, 0.5, 0.4, 0.2 ; *N* = 4096**

Connecting the Disconnected

Hybrid Transition

$$m(t) = \begin{cases} 0 & \text{for} \quad t < t_c \\ m_0 + r(t - t_c)^\beta & \text{for} \quad t \geq t_c \end{cases}$$

g	τ^*	τ
0.1	2.012	2.03 ± 0.04
0.2	2.061	2.08 ± 0.04
0.3	2.111	2.12 ± 0.04
0.4	2.155	2.16 ± 0.04
0.5	2.194	2.18 ± 0.04
0.6	2.231	2.20 ± 0.04
0.7	2.268	2.22 ± 0.04
0.8	2.310	2.25 ± 0.04
0.9	2.364	2.28 ± 0.04

In mean-field the cluster size exponent

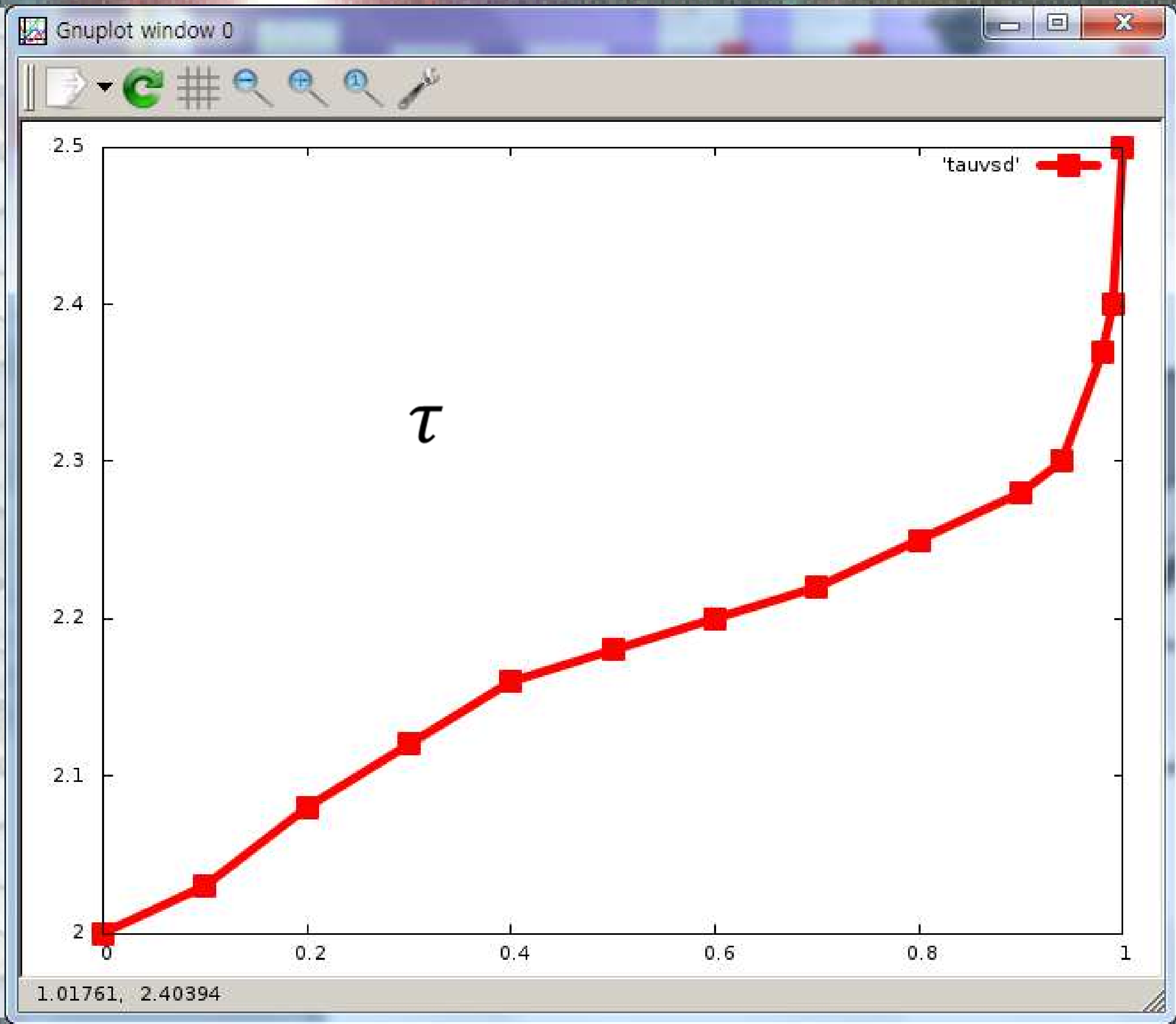
$$2 < \tau < 2.5$$

varies continuously with *g* as:

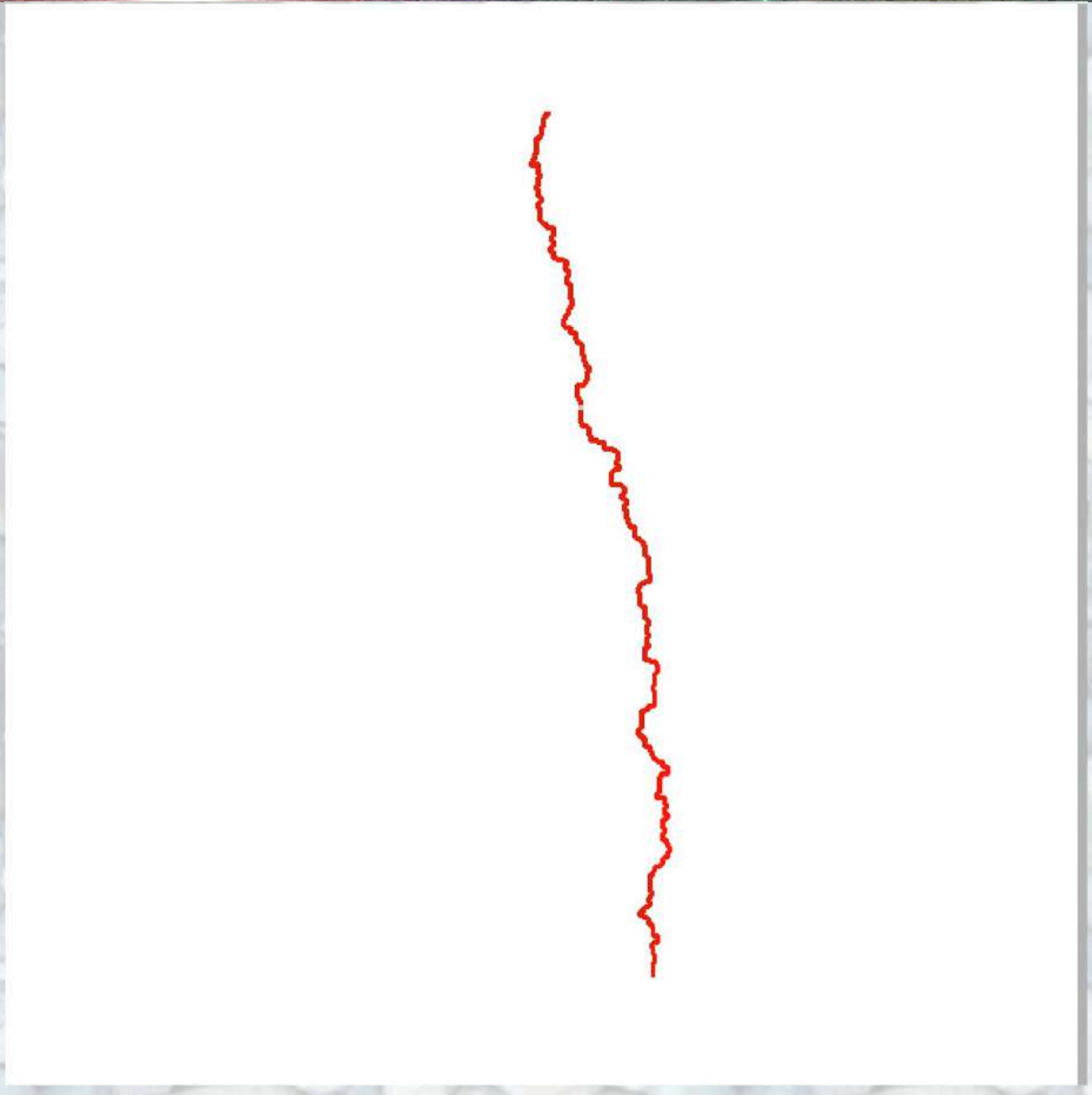
$$\frac{\varsigma(\tau)}{\varsigma(\tau-1)} = \frac{1}{g} - \frac{1}{g+1} \ln\left(\varsigma(\tau-1) \left(\frac{g+1}{2} \right)^{-\left(1+\frac{1}{g}\right)} \right)$$

Y.S. Cho, J.S. Lee, H.J.H., B. Kahng, Phys. Rev. Lett. 116, 025701 (2016)

Connecting the Disconnected

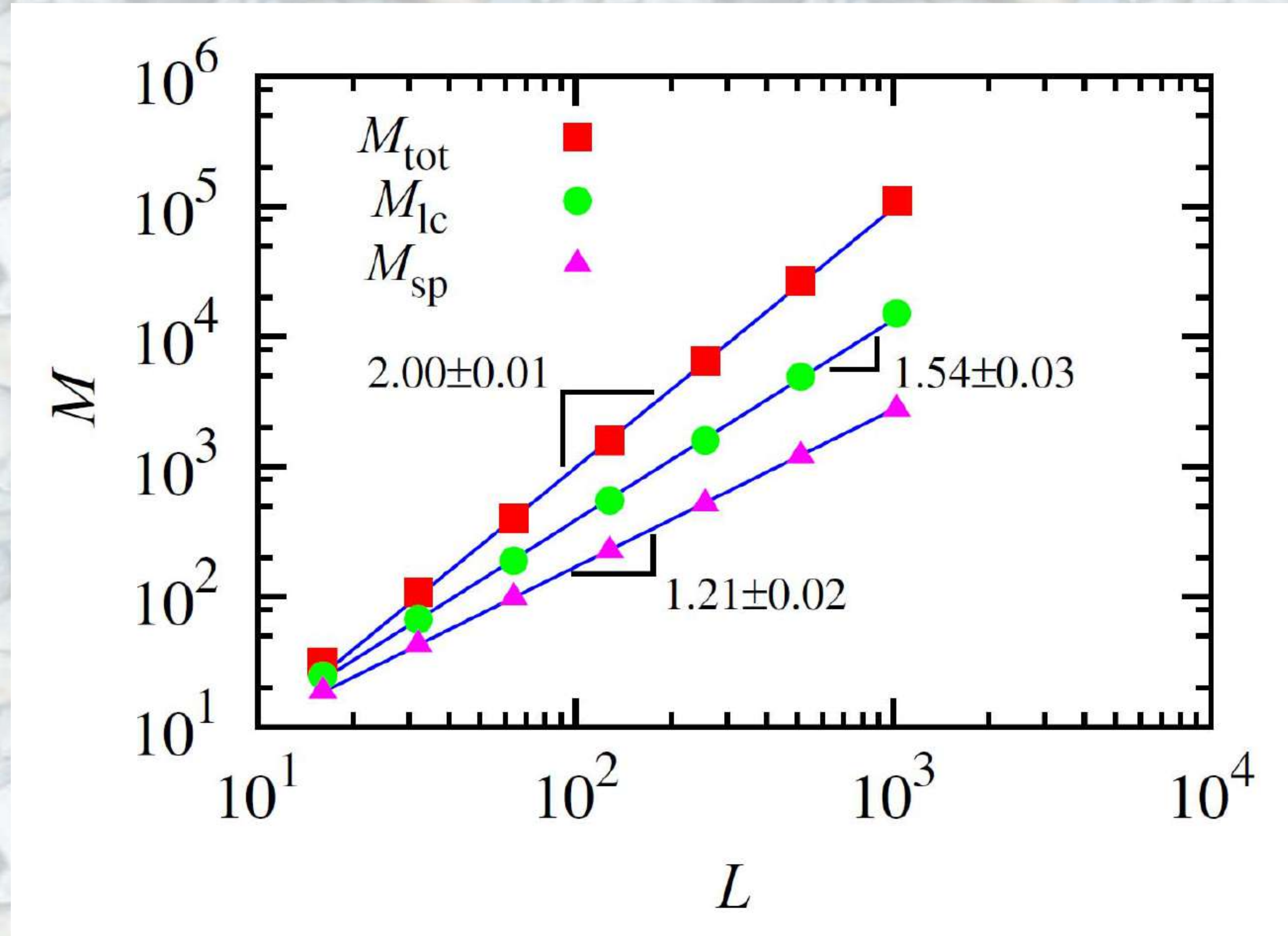


Optimal Path Crack



J. S. Andrade, E.A. Oliveira, A.A. Moreira, HJH, Phys. Rev. Lett. 103, 225503 (2009)

Optimal Path Crack



Optimal Path Crack in strong disorder

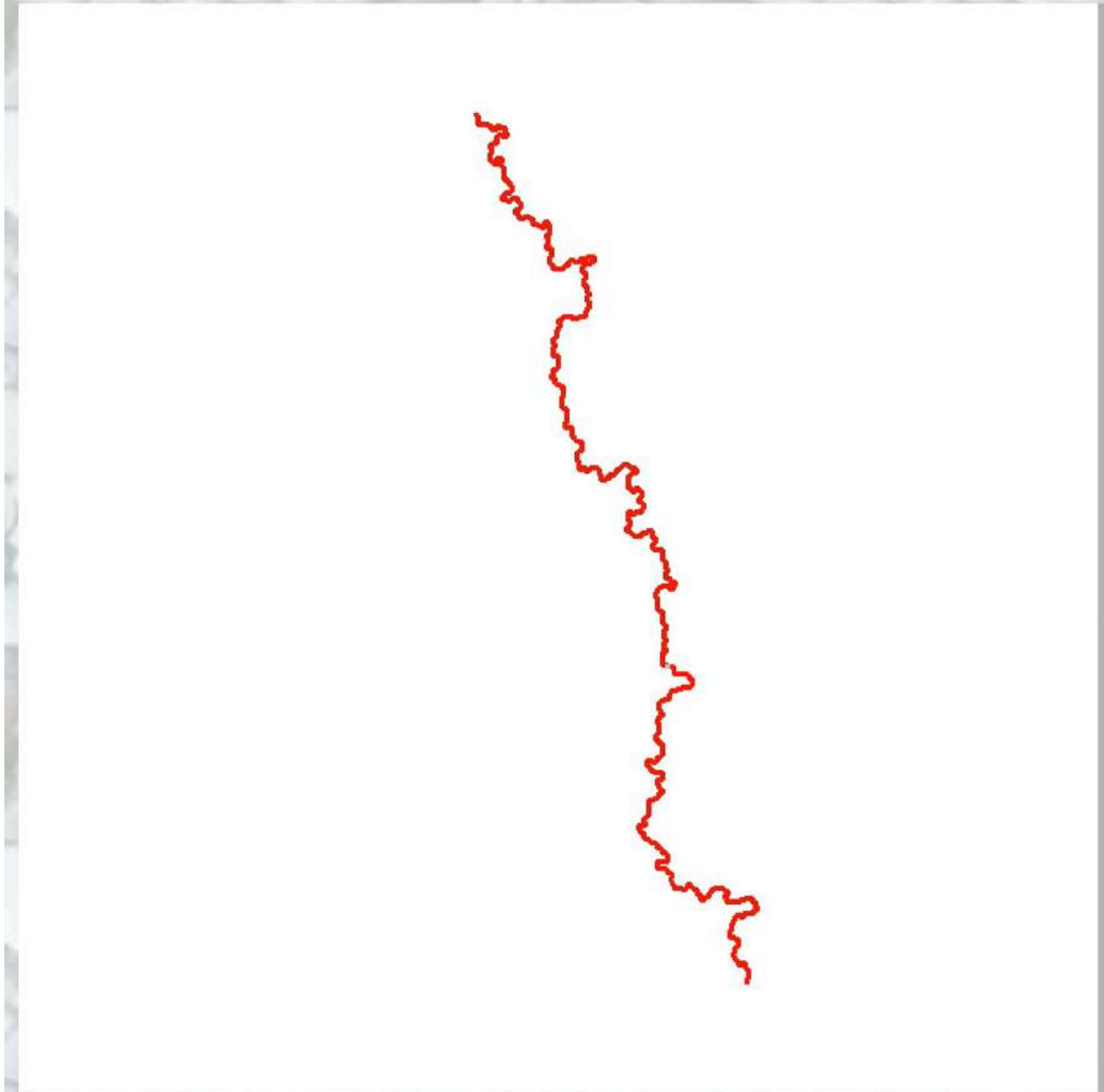
- Consider a random energy landscape with strong disorder, i.e. where the values are distributed randomly according to:

$$p(\varepsilon_i) \propto \frac{1}{\varepsilon_i}$$

- Find the path from top to bottom for which the sum of all energies on this path is minimal. This optimal path has the same fractal dimension as the watershed.
- If one removes from the system the site which on the optimal path had the largest energy, looks for the optimal path in this new system, again removes the site of largest energy and so on, one gets at the end a crack which also has the same fractal dimension.

J. S. Andrade, E.A. Oliveira, A.A. Moreira, HJH, Phys. Rev. Lett. 103, 225503 (2009)

Optimal Path Crack in strong disorder



fractal dimension:

$$d_f = 1.21 \pm 0.02$$

J. S. Andrade, E.A. Oliveira, A.A. Moreira, HJH, Phys. Rev. Lett. 103, 225503 (2009)

Thank you !