Time temperature scaling in complex systems: A new property in nature

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Abstract

Complex systems exhibit different types of scaling, such as **time fluctuation scaling**, characterized by the fact that the dispersion of data in a non-stationary time series (quantified by the variance) satisfies a power-law relationship with respect to the mean of the time series, or **time Theil scaling**, in which the dispersion of data in a diffusive trajectory time series (quantified by the Theil index) satisfies a power-law relationship with respect to the mean of the data through an expression that bears some resemblance to that found in second-order phase transitions.

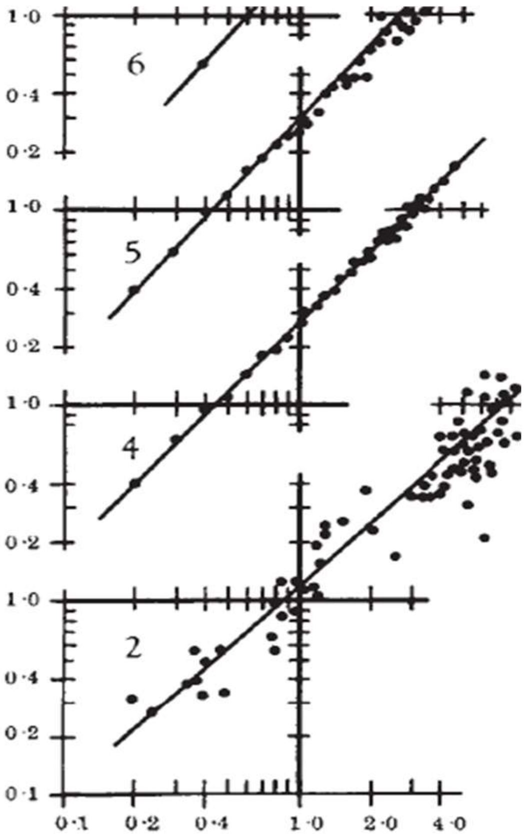
The main goal of this talk is to show the existence in nature of a new scaling relationship, in which the temperature of complex systems, defined as the first absolute central moment, is related by a power law with respect to the mean of the dtime series. This power-law relation between the temperature and the mean, which we have called **time temperature scaling**, is present in financial, economic, epidemic, seismics systems and other kind of systems.

Contents

1. Time fluctuation scaling and time Theil scaling

1.1 Ensamble fluctuation scaling (Taylor law)

L. R. Taylor (1961) discovered this law in ontomological populations where individuals are randomly distributed **[1]**



$$s^2 = am^b$$

	Name	Site and sample	Range of m	Range of s^2	N	a	b
1	Shellfish on seashore, *Tellina tenuis* da Costa, Eulamellibranchiata : Mollusca	Sand, 63 units, various sizes	0·72–45·7	0·49–8·0	5	0·50	0·70
2	European chafer larvæ, *Amphimallon majalis* Raz. (= *Melolontha melolontha* L.), Coleoptera : Insecta	Pasture soil, 25 units, each 1 ft. sq.	0·20–9·72	0·26–14·93	75	1·15	1·07
3	Flying insects, various orders : Insecta	Open air, 16–104 units aerial density	1·9–238	1·7–606	24	1·0	1·17
4	Wireworms, *Agriotes* spp. mainly *obscurus*, Coleoptera : Insecta	Grassland soil 20 units, 4 in. cores	0·20–4·65	0·40–17·80	2,272	2·75	1·19
5	Wireworms, *Agriotes* as above	Arable land soil 20 units, 4 in. cores	0·20–4·65	0·39–22·50	525	2·85	1·26
6	Wireworms, *Limonius* spp., Coleoptera : Insecta	Arable land soil, 175 units, each 1 ft. sq.	0·39–10·89	0·58–60·42	24	2·0	1·33

x-axis: Logarithm of mean (m). y-axis: Logarithm of variance (s^2). Figure and table taken of **[1]**.

[1] L. R. Taylor, “Aggregation, variance and the mean”, Nature 189 (1961) 1-28.

1. Time fluctuation scaling and time Theil scaling

1.2 Time fluctuation scaling

Variance and mean satisfies a power-law relation of the form

$$\Xi_2^{(H)}(t) = K\left[\Upsilon_1^{(H)}(t)\right]^{\alpha(t)}$$

This property is present in non-stationary time series, for example, in systems of

Ecology **[2]** J. W. Kirchner, X. Feng and C, Neal, “Fractal stream chemistry and its implication for contaminant transport in catchments”, Nature 403 (2000) 524-527; M. Tokeshi, “On the mathematical basis of the variance-mean power relationship”, Researches on Population Ecology37 (1995) 43-48.

Neurosciences **[3]** K. Linkenkaer-Hansen, V. V. Nikouline, J. M. Palvas and R. J. Ilmoniemi, “Long-range temporal correlations and scaling behavior in human brain oscillations”, The Journal of Neuroscience 21 (2001) 1370-1377.

Turbulence and percolation **[4]** G.Paladin and A. Vulpiani, “Anomalous scaling laws in multifractal objects”, Physics Reports 156 (1987) 147-225.

Website visits, river networks and highway traffic **[5]** M. Argollo de Menezes and A. L. Barabási, “Fluctuations in network dynamics”, Physical Review Letters 92 (2004) 028701.

Financial markets **[6]** P. Gopiksishnan, V. Plerou, L. A, Nunes Amaral, M. Meyer and H. E. Stanley, “Scaling of the distribution of fluctuations of financial market indices”, Physical Review E 60 (1999) 5305-5316.

1. Time fluctuation scaling and time Theil scaling

1.2 Time fluctuation scaling

Path integral formalism can be used to describe evolution of mean **[7,8]**

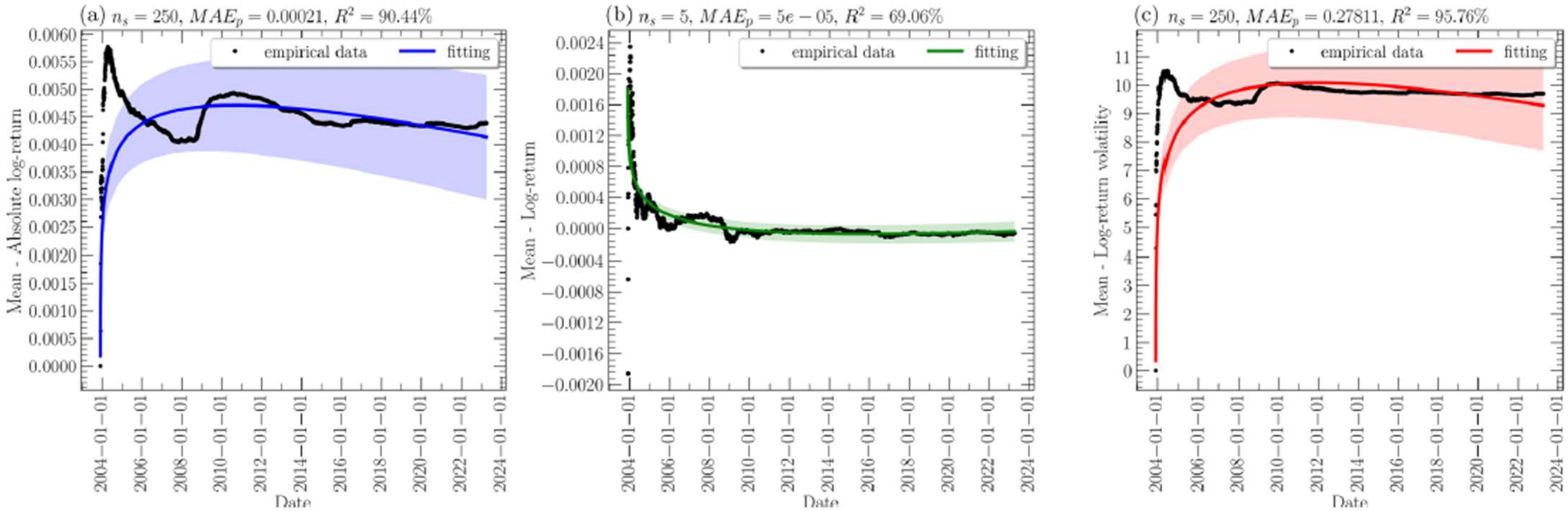


FIG. 4. Evolution of the mean of the time series of the U.K. pound sterling to U.S. dollar exchange rate measured daily from December 2, 2003, to April 11, 2023, taking the average of the total accumulated data for each optimal window size N_s: (a) absolute log-returns time series ($N_s = 250$), (b) log-returns time series ($N_s = 5$), and (c) volatilities of the log-returns time series ($N_s = 250$). In all cases, the shaded region corresponds to a confidence interval constructed using the uncertainty $\Delta\underline{\gamma}$ of the regression parameters.

Figure taken of **[8]**.

[7] F. S. Abril and C. J. Quimbay, "Temporal fluctuation scaling in non-stationary time series using the path integral formalism", Physics Review E 103 (2021) 042126.
[8] F. S. Abril and C. J. Quimbay, "Evolution of the temporal fluctuation scaling exponent in non-stationary time series using supersymmetric theory of stochastic dynamics", Physics Review E 109 (2024) 024112.

1. Time fluctuation scaling and time Theil scaling

1.2 Time fluctuation scaling

Path integral formalism can be used to describe evolution of variance **[7,8]**

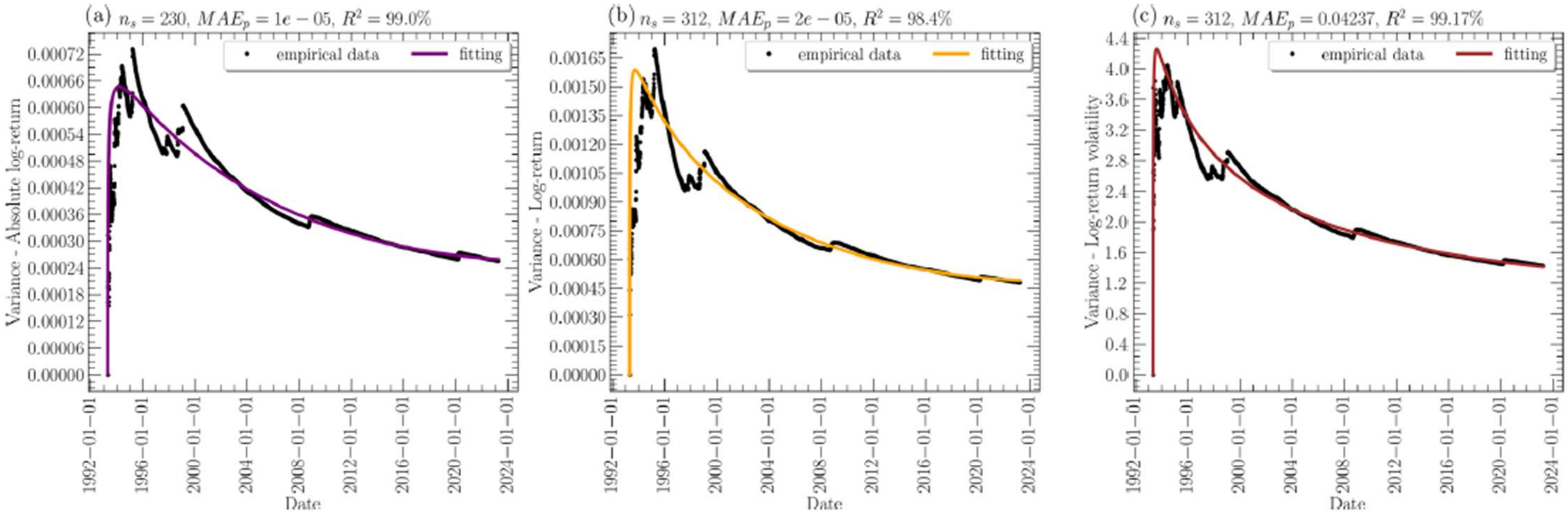


FIG. 7. Evolution of the variance of the time series of the Sao Paulo Stock Exchange measured daily from April 28, 1993, to April 11, 2023, taking the average of the total accumulated data for each optimal window size N_s: (a) absolute log-returns time series ($N_s = 230$), (b) log-returns time series ($N_s = 312$), and (c) volatilities of the log-returns time series ($N_s = 312$). In all cases, the shaded region corresponds to a confidence interval constructed using the uncertainty $\Delta\gamma$ of the regression parameters.

Figure taken of **[8]**.

[7] F. S. Abril and C. J. Quimbay, "Temporal fluctuation scaling in non-stationary time series using the path integral formalism", Physics Review E 103 (2021) 042126.

[8] F. S. Abril and C. J. Quimbay, "Evolution of the temporal fluctuation scaling exponent in non-stationary time series using supersymmetric theory of stochastic dynamics", Physics Review E 109 (2024) 024112.

1. Time fluctuation scaling and time Theil scaling

1.2 Time fluctuation scaling

Path integral formalism can be used to describe evolution of the exponent of the power-law relation **[7,8]**

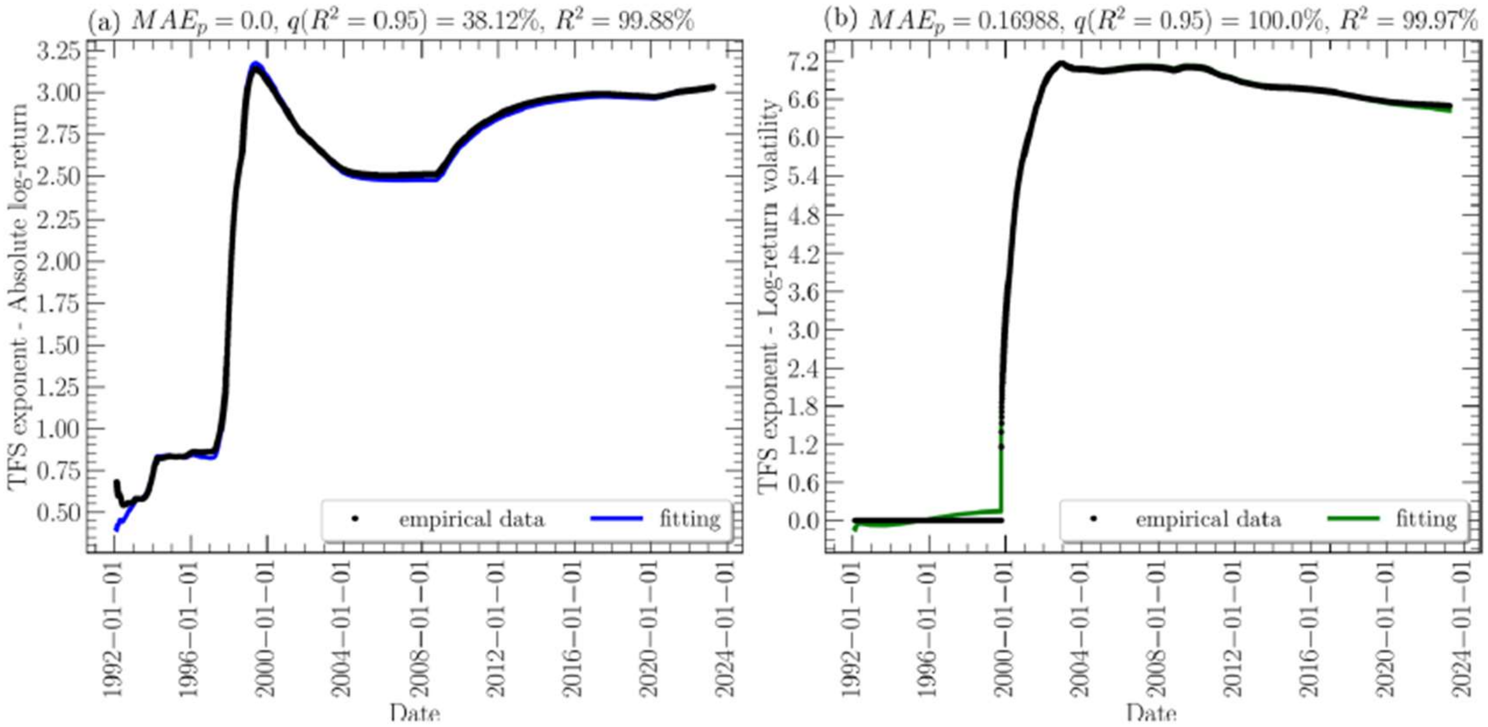


FIG. 10. Evolution of the temporal fluctuation scaling exponent $\alpha_{\mathrm{TFS}}(t)$ of the time series of the Dow Jones Industrial Average measured daily from January 3, 1992, to April 11, 2023: (a) absolute log-returns time series and (b) volatilities of the log-returns time series.

Figure taken of **[8]**.

[7] F. S. Abril and C. J. Quimbay, "Temporal fluctuation scaling in non-stationary time series using the path integral formalism", Physics Review E 103 (2021) 042126.

[8] F. S. Abril and C. J. Quimbay, "Evolution of the temporal fluctuation scaling exponent in non-stationary time series using supersymmetric theory of stochastic dynamics", Physics Review E 109 (2024) 024112.

1. Time fluctuation scaling and time Theil scaling

1.3 Time Theil scaling

Theil index (T) is defined as
$$T(t) = -\sum_k \frac{x_{\mathrm{TS}}^{(k)}(t)}{\bar{x}(t)M} \ln\left[\frac{x_{\mathrm{TS}}^{(k)}(t)}{\bar{x}(t)M}\right],$$

where $x_{\mathrm{TS}}^{(k)}(t)$ is the kth value of the diffusive trajectory time series at time t, $\bar{x}(t)$ is the mean of the diffusive trajectory time series at time t and $M = N - t + 1$ is the total data number of diffusive trajectory at time t.

The diffusive trajectory time series is obtained from the original time series applying the diffusion algorithm **[9]**

$$x_{\mathrm{TS}}^{(s)}(t) = \sum_{j=1}^{t} \mathrm{TS}_{j+s},$$

with $s = 0, 1, \ldots, N - t$. The last expression defines in each time step a subseries of data in such a way that for $t = 1$ corresponds to the original time series, for $t = 2$ corresponds to the subseries obtained by the sum of two successive terms of the original time series, and for $t = N$, the subseries now has only one data point which corresponds to the sum of the N data of the original time series.

[9] N. Scafetta and P. Grigolini, "Scaling dectection in time series: Diffusion entropy analysis", Physics Review E 66 (2002) 036130.

1. Time fluctuation scaling and time Theil scaling

1.3 Time Theil scaling

We find that diffusive trajectory time series doesn't present the property of time fluctuation scaling, implying that relationship of power law of the form

$$\Xi(t) = K|\Upsilon(t)|^{\alpha}$$

is not satisfied.

However, we find for 48 diffusive trajectories time series, there exists a power-law relation of the form **[10]**

$$\frac{T(t)}{T(1)} = K_1 \left| 1 - \frac{\Upsilon(t)}{\Upsilon_M} \right|^{\beta},$$

where $\Upsilon_M = \max\{\Upsilon(j)|1 \leqslant j \leqslant N\}$ and $T(1)$ is the maximum value of the Theil index throughout the different time steps taken which corresponds to the first time step (t=1), K_1 is the TTS constant and β is the TTS exponent. We call this power-law relation involving T as temporal Theil scaling (TTS).

[10] F. S. Abril and C. J. Quimbay, "Temporal Theil scaling in diffusive trajectory time series", Physics Review E 106 (2022) 014117.

1. Time fluctuation scaling and time Theil scaling

1.3 Time Theil scaling

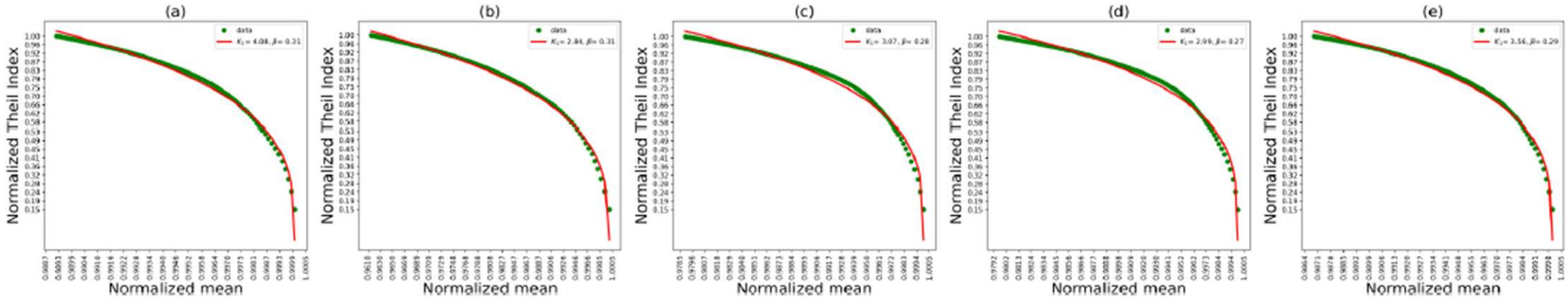


FIG. 18. Normalized Theil index $T(t)/T(1)$ as a function of the normalized mean $\Upsilon(t)/\Upsilon_M$ for the diffusive trajectory time series of volatility of the: (a) Colombian peso-dollar (COP-USD), (b) bitcoin-dollar (BTC-USD), (c) euro-dollar (EUR-USD), (d) pound sterling-dollar (GBP-USD), and (e) dollar-yen (USD-JPY) currencies taking 100 time steps. Red line: fit with Eq. (10). Green line: Empirical data.

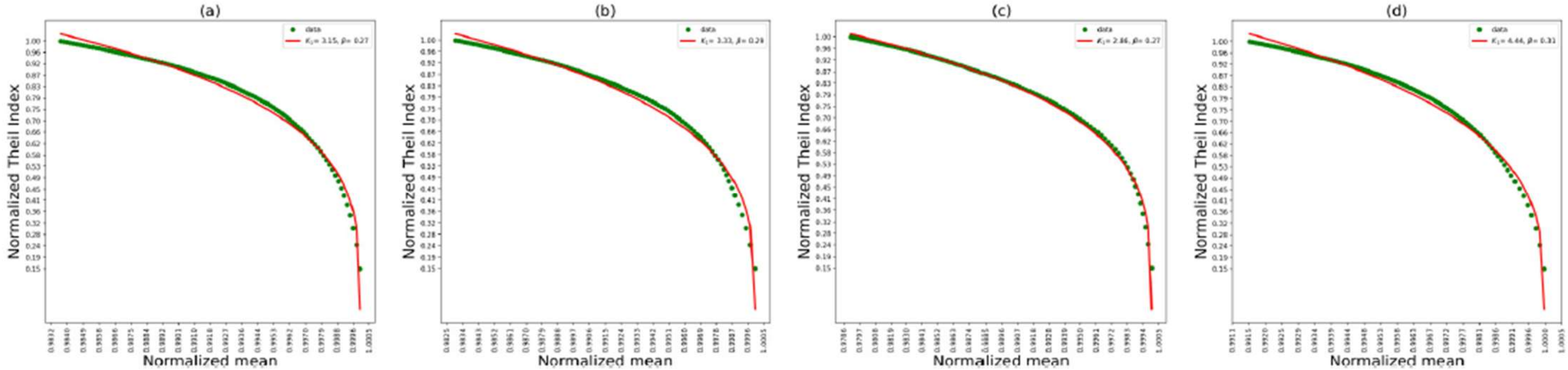


FIG. 19. Normalized Theil index $T(t)/T(1)$ as a function of the normalized mean $\Upsilon(t)/\Upsilon_M$ for the diffusive trajectory time series of volatility of the: (a) silver, (b) gold, (c) crude oil commodities, and (d) the treasure yield of United States taking 100 time steps. Red line: fit with Eq. (10). Green line: Empirical data.

Figures taken of **[10]** F. S. Abril and C. J. Quimbay, "Temporal Theil scaling in diffusive trajectory time series", Physics Review E 106 (2022) 014117.

1. Time fluctuation scaling and time Theil scaling

1.3 Time Theil scaling

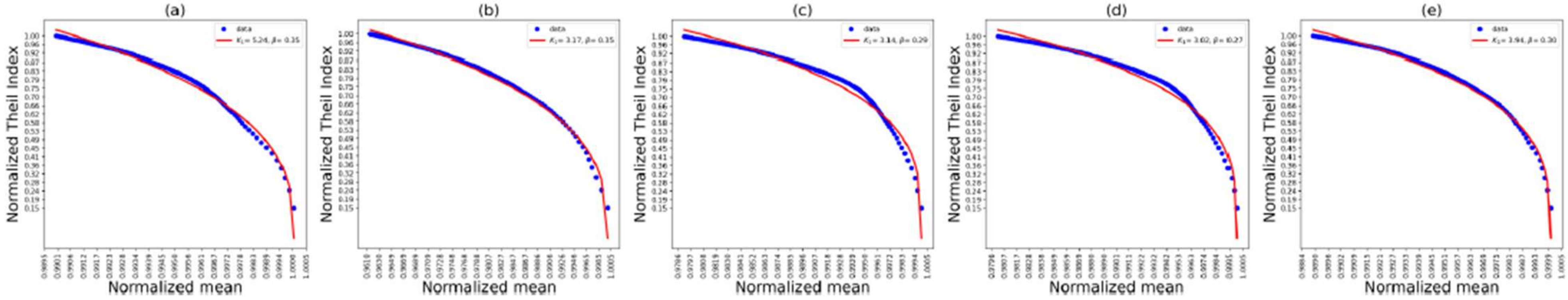


FIG. 22. Normalized Theil index $T(t)/T(1)$ as a function of the normalized mean $\Upsilon(t)/\Upsilon_M$ for the diffusive trajectory time series of absolute log-return of the: (a) Colombian peso-dollar (COP-USD), (b) bitcoin-dollar (BTC-USD), (c) euro-dollar (EUR-USD), (d) pound sterling-dollar (GBP-USD), and (e) dollar-yen (USD-JPY) currencies taking 100 time steps. Red line: fit with Eq. (10). Blue line: Empirical data.

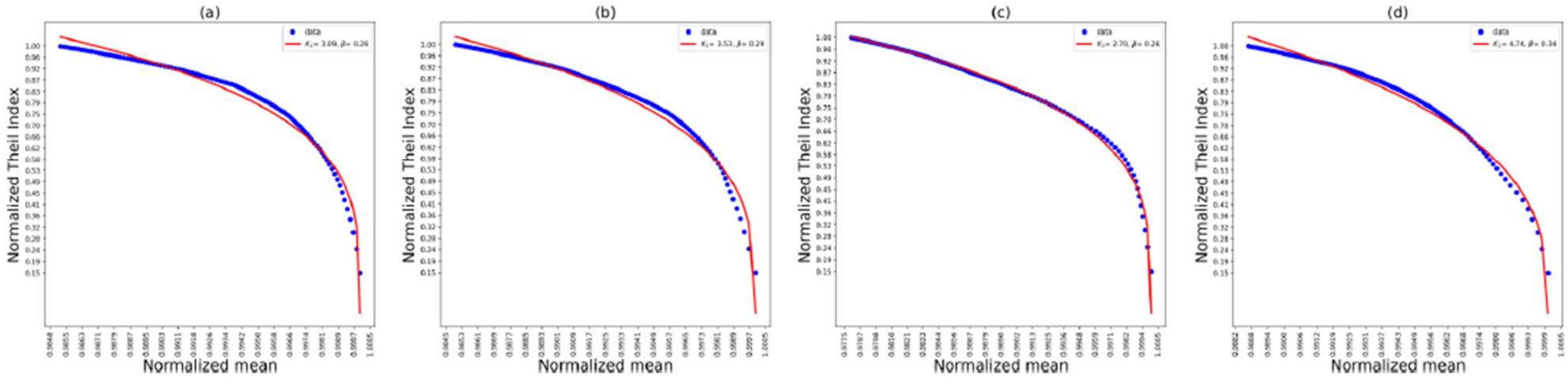


FIG. 23. Normalized Theil index $T(t)/T(1)$ as a function of the normalized mean $\Upsilon(t)/\Upsilon_M$ for the diffusive trajectory time series of absolute log-return of the: (a) silver, (b) gold, (c) crude oil commodities, and (d) the treasure yield of United States taking 100 time steps. Red line: fit with Eq. (10). Blue line: Empirical data.

Figures taken of **[10]** F. S. Abril and C. J. Quimbay, “Temporal Theil scaling in diffusive trajectory time series”, Physics Review E 106 (2022) 014117.

1. Time fluctuation scaling and time Theil scaling

1.3 Time Theil scaling

TABLE II. K_1 and β adjustment parameters of the TTS [see Eq. (10)] for the diffusive trajectory time series of volatility of the Nikkei 225, S&P 500, DAX, MOEX, IBEX 35, NASDAQ, BOVESPA, COLCAP, CAC 40, AEX, and RTS stock indexes, the Colombian peso-dollar (COP-USD), bitcoin-dollar (BTC-USD), euro-dollar (EUR-USD), pound sterling-dollar (GBP-USD), dollar-yen (USD-JPY) currencies, the silver, gold, crude oil commodities, the treasure yield of United States, the temperature and precipitation in Bogota, Colombia, and the daily cases and deaths of COVID-19 in the United States taking 100 time steps.

Time Series	K_1	β	GAE(%)	χ^2
Nikkei 225	4,0981±0.0996	0.2835±0.0044	3,0287	0.1967
S&P 500	2,9111±0.0482	0.2847±0.0038	2,4218	0.1781
DAX	4,1777±0.0875	0.3263±0.0043	2,3262	0.1717
MOEX	2,6706±0.0568	0.3362±0.0062	3,8149	0.2475
IBEX 35	3,6693±0.0790	0.2961±0.0043	2,8372	0.1907
NASDAQ	4,1137±0.0891	0.2901±0.0040	2,5263	0.1807
BOVESPA	3,2030±0.0653	0.2694±0.0041	2,9554	0.1996
COLCAP	2,7757±0.0549	0.2955±0.0049	3,3215	0.2114
CAC 40	3,7844±0.0775	0.3012±0.0041	2,5474	0.1799
AEX	3,7360±0.0757	0.2973±0.0041	2,5306	0.1761
RTS	3,8714±0.0980	0.3281±0.0055	3,3145	0.2288
USD-COP	4,0795±0.0992	0.3055±0.0047	3,0563	0.1945
BTC-USD	2,8411±0.0501	0.3146±0.0046	2,7518	0.1941
EUR-USD	3,0711±0.0724	0.2838±0.0052	3,7530	0.2410
GBP-USD	2,9894±0.0674	0.2743±0.0049	3,6527	0.2281
USD-JPY	3,5605±0.0798	0.2875±0.0045	3,0817	0.2031
Silver	3,1480±0.0714	0.2724±0.0047	3,4864	0.2279
Gold	3,3255±0.0797	0.2893±0.0051	3,5291	0.2392
Crude Oil	2,8565±0.0483	0.2683±0.0037	2,4248	0.1933
Treasure Yield U.S.	4,4410±0.1226	0.3064±0.0051	3,3643	0.2109
Bogota Temperature	3,0882±0.0604	0.2995±0.0045	2,9192	0.1938
Bogota precipitation	2,0561±0.0259	0.2709±0.0038	2,6672	0.1866
Covid Cases USA	1,5159±0.0108	0.2653±0.0032	1,7125	0.1614
Covid Deaths USA	1,7672±0.0217	0.3081±0.0051	3,2799	0.2132

Table taken of **[10]** F. S. Abril and C. J. Quimbay, “Temporal Theil scaling in diffusive trajectory time series”, Physics Review E 106 (2022) 014117.

1. Time fluctuation scaling and time Theil scaling

1.3 Time Theil scaling

TABLE III. K_1 and β adjustment parameters of the TTS [see Eq. (10)] for the diffusive trajectory time series of absolute log-return of the Nikkei 225, S&P 500, DAX, MOEX, IBEX 35, NASDAQ, BOVESPA, COLCAP, CAC 40, AEX, and RTS stock indexes, the Colombian peso-dollar (COP-USD), bitcoin-dollar (BTC-USD), euro-dollar (EUR-USD), pound sterling-dollar (GBP-USD), dollar-yen (USD-JPY) currencies, the silver, gold, crude oil commodities, the treasure yield of United States, the temperature and precipitation in Bogota, Colombia, and the daily cases and deaths of COVID-19 in the United States taking 100 time steps.

Time Series	K_1	β	GAE(%)	χ^2
Nikkei 225	4,3937±0.1193	0.2964±0.0049	3,2884	0.2048
S&P 500	3,2044±0.0553	0.3064±0.0040	2,2623	0.1739
DAX	5,3874±0.1201	0.3984±0.0048	1,8793	0.1591
MOEX	3,6795±0.1137	0.5353±0.0112	4,4052	0.2732
IBEX 35	4,2887±0.0959	0.3296±0.0045	2,5600	0.1750
NASDAQ	4,2715±0.0888	0.3035±0.0039	2,2074	0.1716
BOVESPA	2,9590±0.0538	0.2545±0.0037	2,6853	0.1902
COLCAP	3,0303±0.0733	0.3413±0.0065	3,6925	0.2341
CAC 40	4,5257±0.0951	0.3444±0.0043	2,1569	0.1643
AEX	4,5630±0.1012	0.3374±0.0044	2,3420	0.1678
RTS	6,4680±0.2407	0.4876±0.0089	3,7416	0.2464
USD-COP	5,2412±0.1630	0.3531±0.0060	3,3283	0.2004
BTC-USD	3,1673±0.0610	0.3480±0.0051	2,7436	0.1950
EUR-USD	3,1365±0.0903	0.2872±0.0063	4,5848	0.3000
GBP-USD	3,0152±0.0774	0.2736±0.0055	4,1862	0.2573
USD-JPY	3,9384±0.0927	0.2980±0.0046	2,9871	0.1936
Silver	3,0878±0.0858	0.2594±0.0056	4,4626	0.2953
Gold	3,5263±0.1023	0.2937±0.0060	4,1470	0.2941
Crude Oil	2,6979±0.0464	0.2594±0.0038	2,4971	0.2081
Treasure Yield U.S.	4,7363±0.1385	0.3398±0.0058	3,3689	0.2041
Bogota Temperature	2,9674±0.0587	0.2972±0.0047	3,1071	0.1995
Bogota precipitation	2,1271±0.0291	0.2847±0.0042	2,8777	0.1914
Covid Cases USA	1,3988±0.0090	0.2676±0.0033	1,9184	0.1606
Covid Deaths USA	1,7903±0.0235	0.3339±0.0058	3,4858	0.2257

Table taken of **[10]** F. S. Abril and C. J. Quimbay, “Temporal Theil scaling in diffusive trajectory time series”, Physics Review E 106 (2022) 014117.

2. Temperature of complex systems

For a complex system with N agents, we define the temperature of the complex system (T_{cs}) as

$$T_{cs} = \sum_{i=1}^{m} p_i |x_i - \bar{x}|, \tag{1}$$

where m is the range number and $\bar{x}$ is the mean of the quantity given by

$$\bar{x} = \mu = \sum_{i=1}^{m} p_i x_i, \tag{2}$$

being p_i the probability to find an agent with quantity in the range x_i

$$p_i = \frac{n_i}{N} \tag{3}$$

2. Temperature of complex systems

If $m = N$, then $n_i = 1$ and $p_i = \frac{1}{N}$, in such a way that T_{cs} is written as

$$T_{cs} = \frac{1}{N}\sum_{l=1}^{N}|x_l - \bar{x}|. \tag{4}$$

We observe that T_{cs} is defined as the **first absolute central moment**.

First absolute central moment is a measure of the dispersion (fluctuations) which is different to the standard deviation (σ).

In image processing, it is used as a filter to measure the variability of gray levels, helping to detect discontinuities (edges) in images.

This quantity is particularly useful for measuring the "mean absolute deviation," which can offer a more robust estimate of dispersion compared to the variance or standard deviation in distributions with heavy tails.

3. Time temperature scaling in different systems

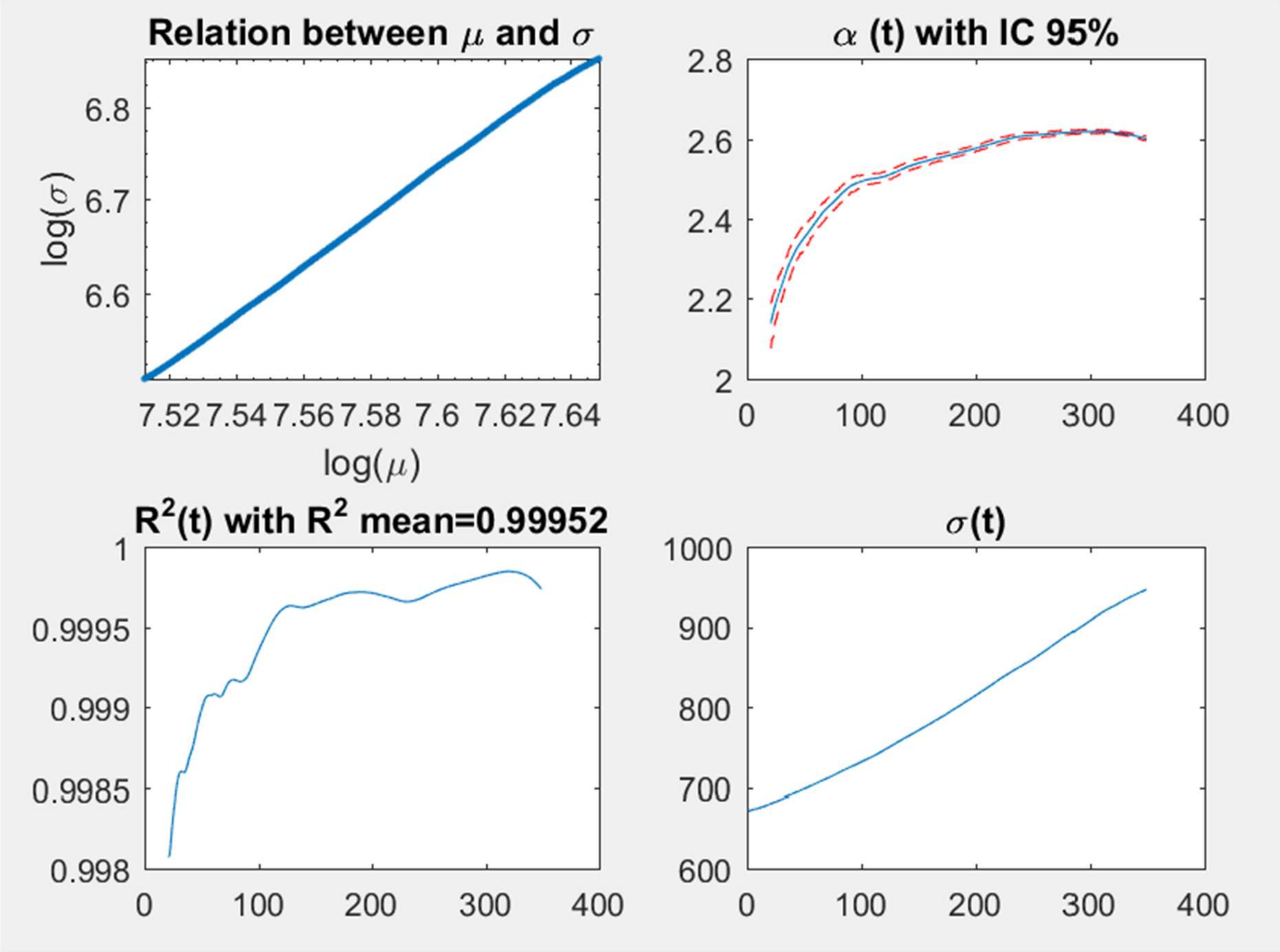
Analysis of time series in financial, economic, epidemic and seismic systems lead us to find a power-law relation between T_{cs} and $\bar{x}$ of the form

$$T_{cs}(t) = K \mid \bar{x}(t) \mid^{\alpha(t)}, \tag{5}$$

that we call **time temperature scaling**.

3. Time temperature scaling in different systems

SP500 index (1/4/2006 – 3/10/2022) =4073 data - Mobile window of 3725 data



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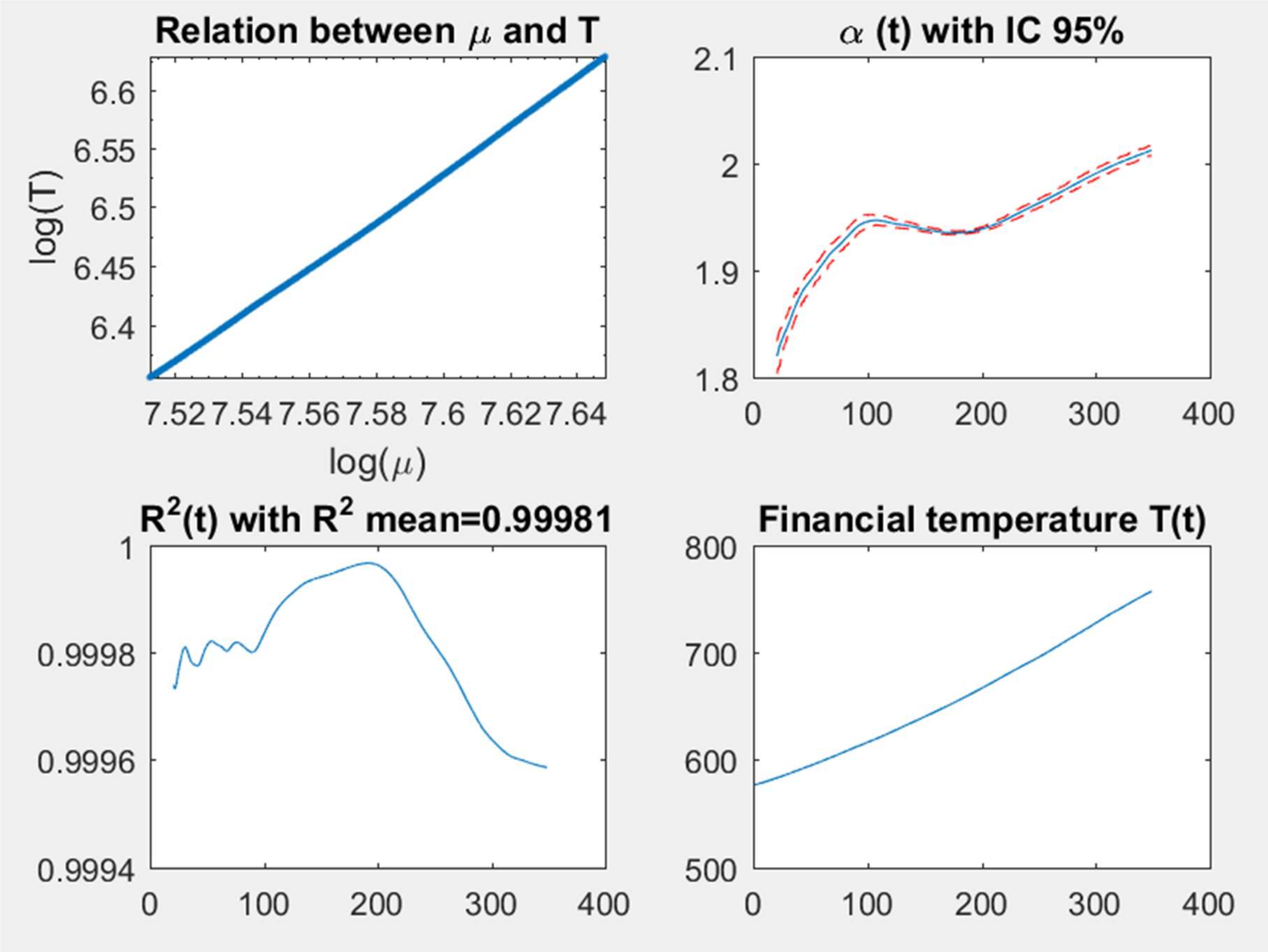
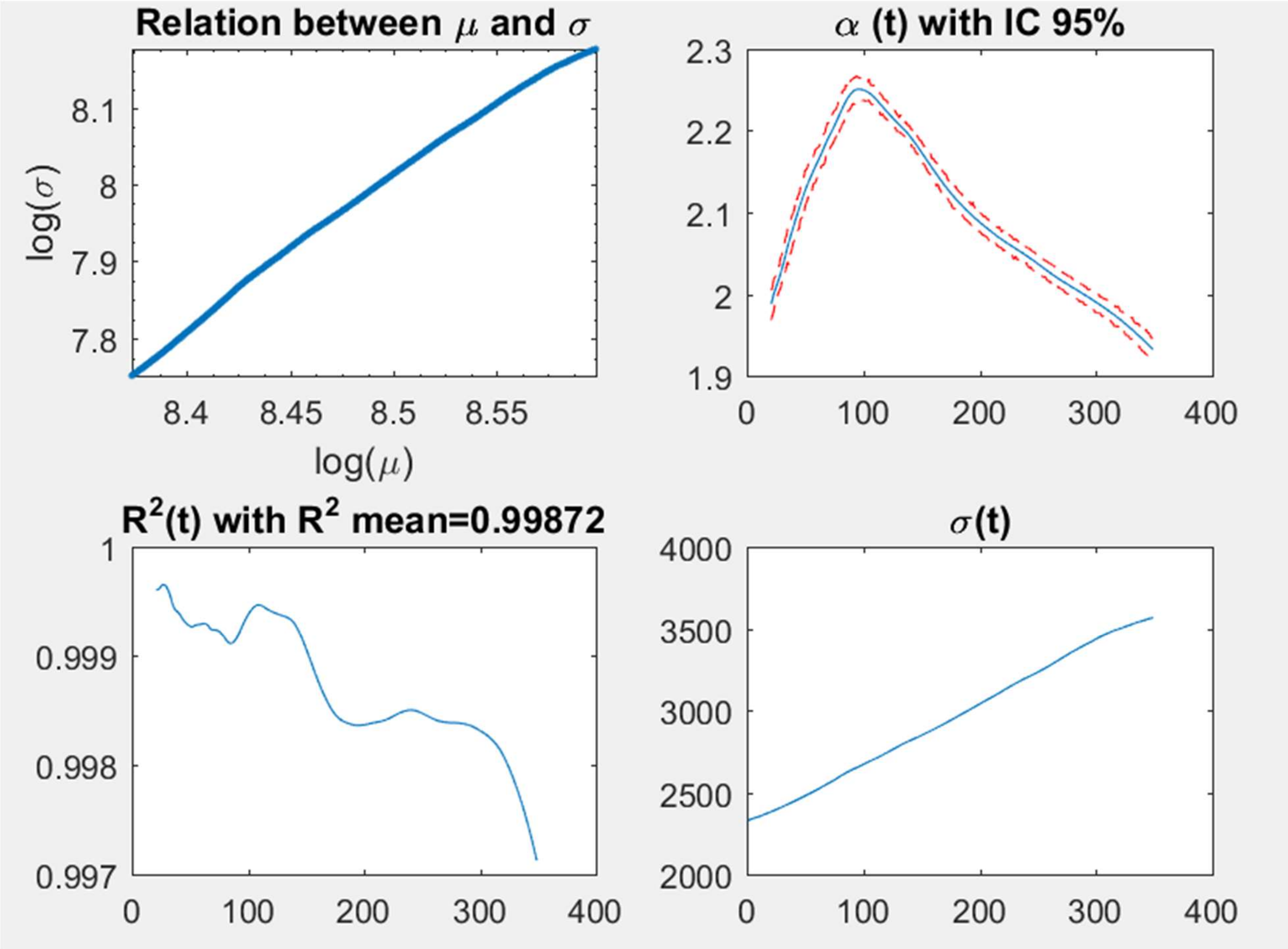


3. Time temperature scaling in different systems

Nasdaq index (1/4/2006– 3/9/2022) =4073 data - Mobile window of 3725 data



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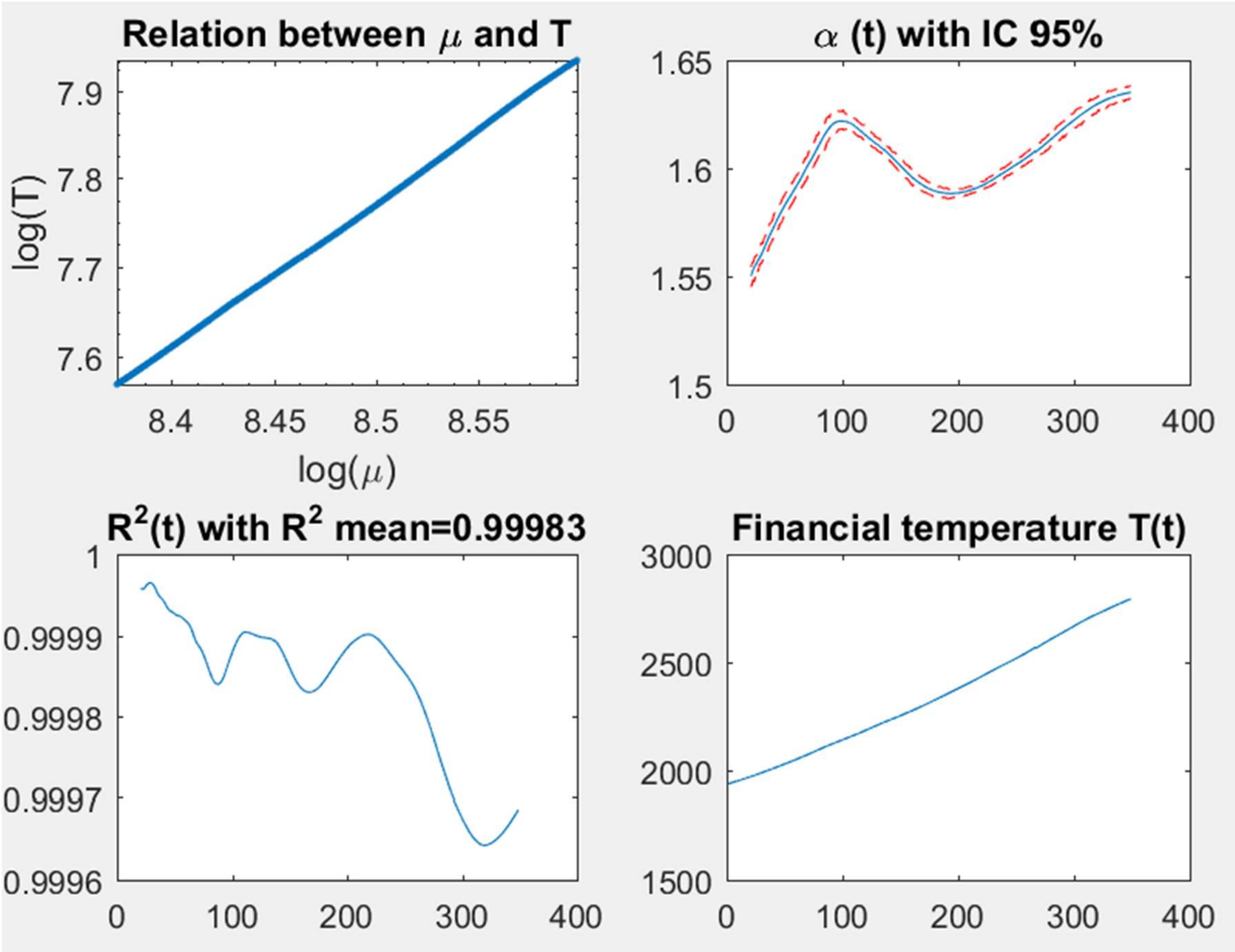
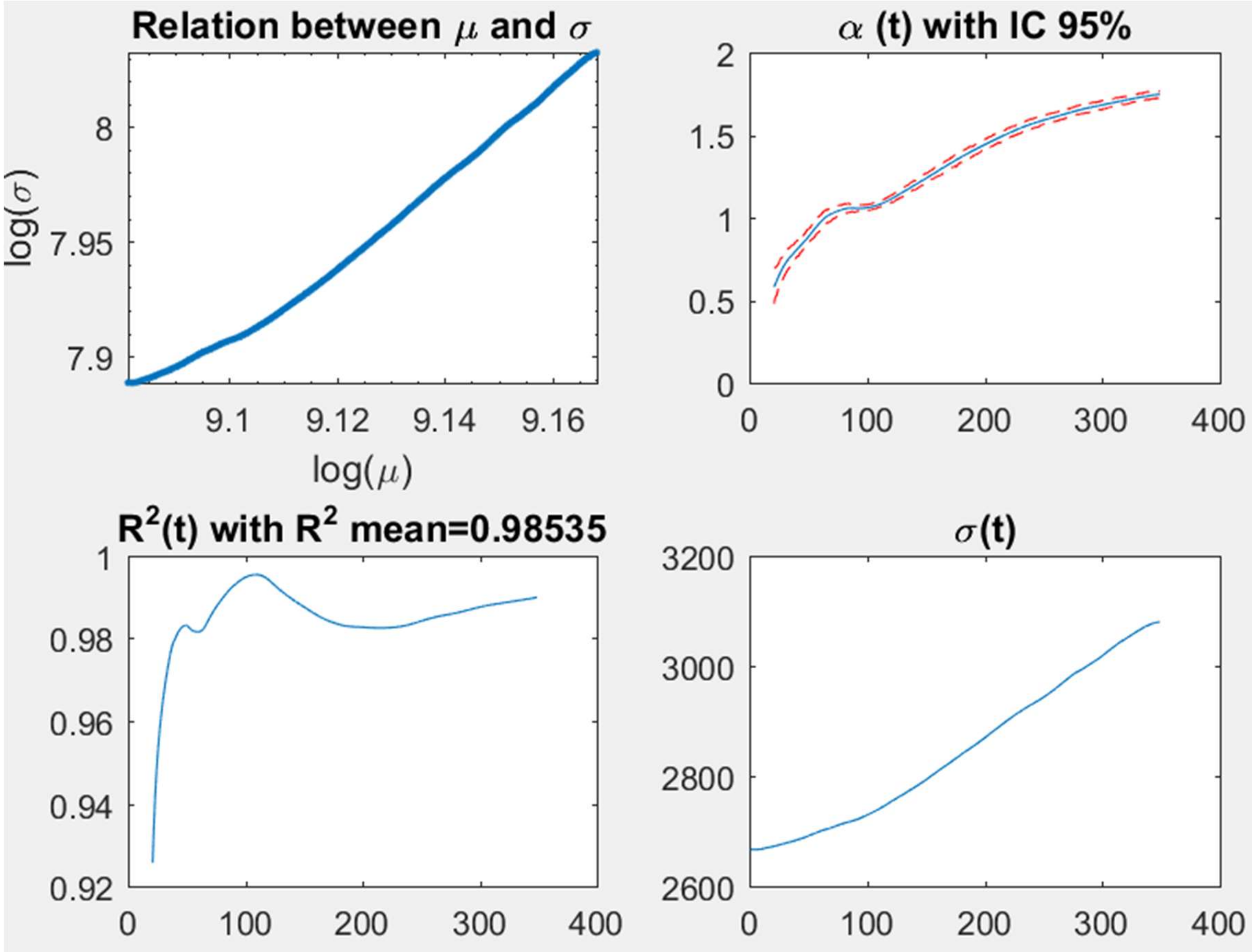


3. Time temperature scaling in different systems

DAX40 index (2/13/2006– 3/9/2022) =4073 data - Mobile window of 3725 data



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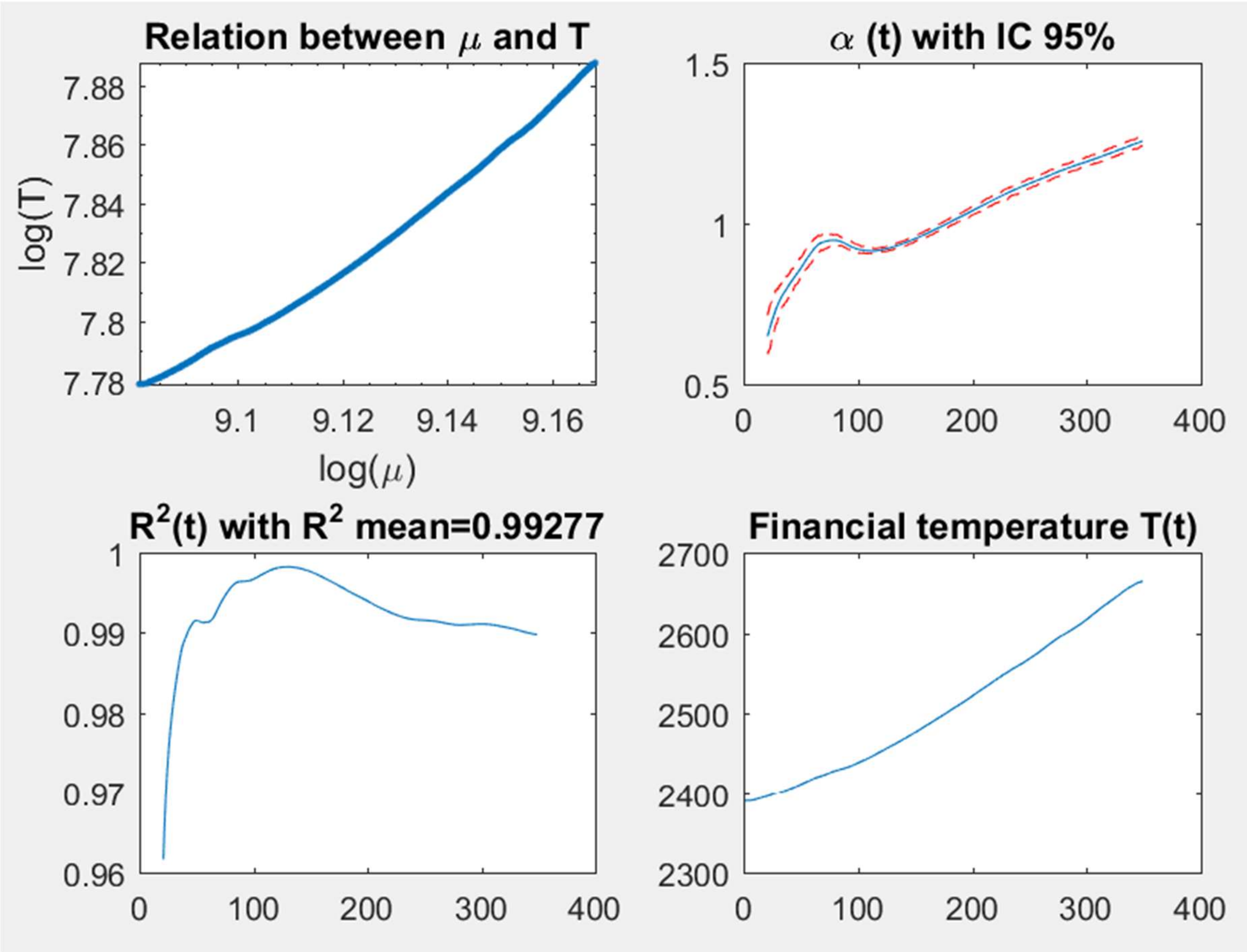
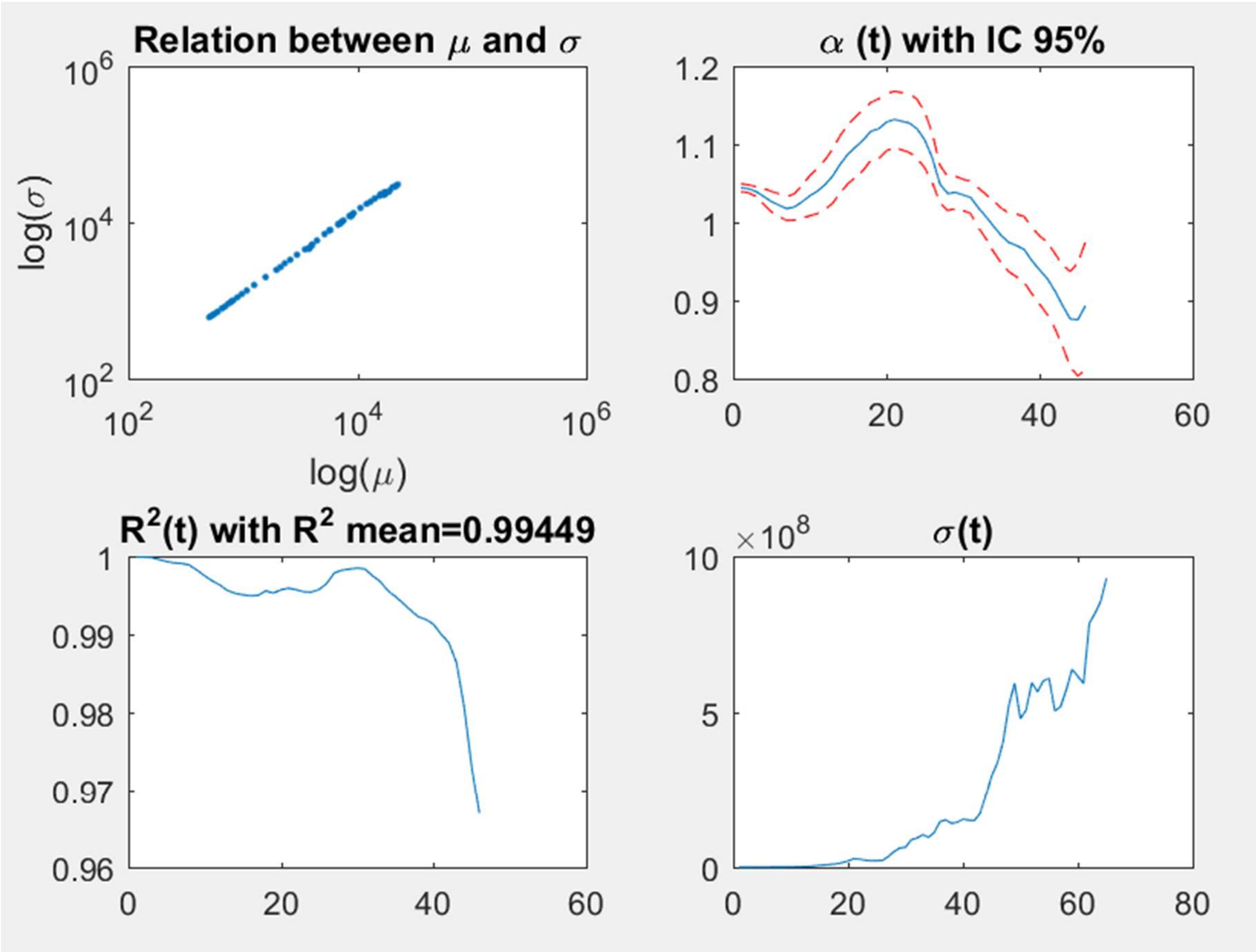


3. Time temperature scaling in different systems

Gross domestic product per capita of 110 countries (1960-2024) = 65 years
Mobile window of 20 years



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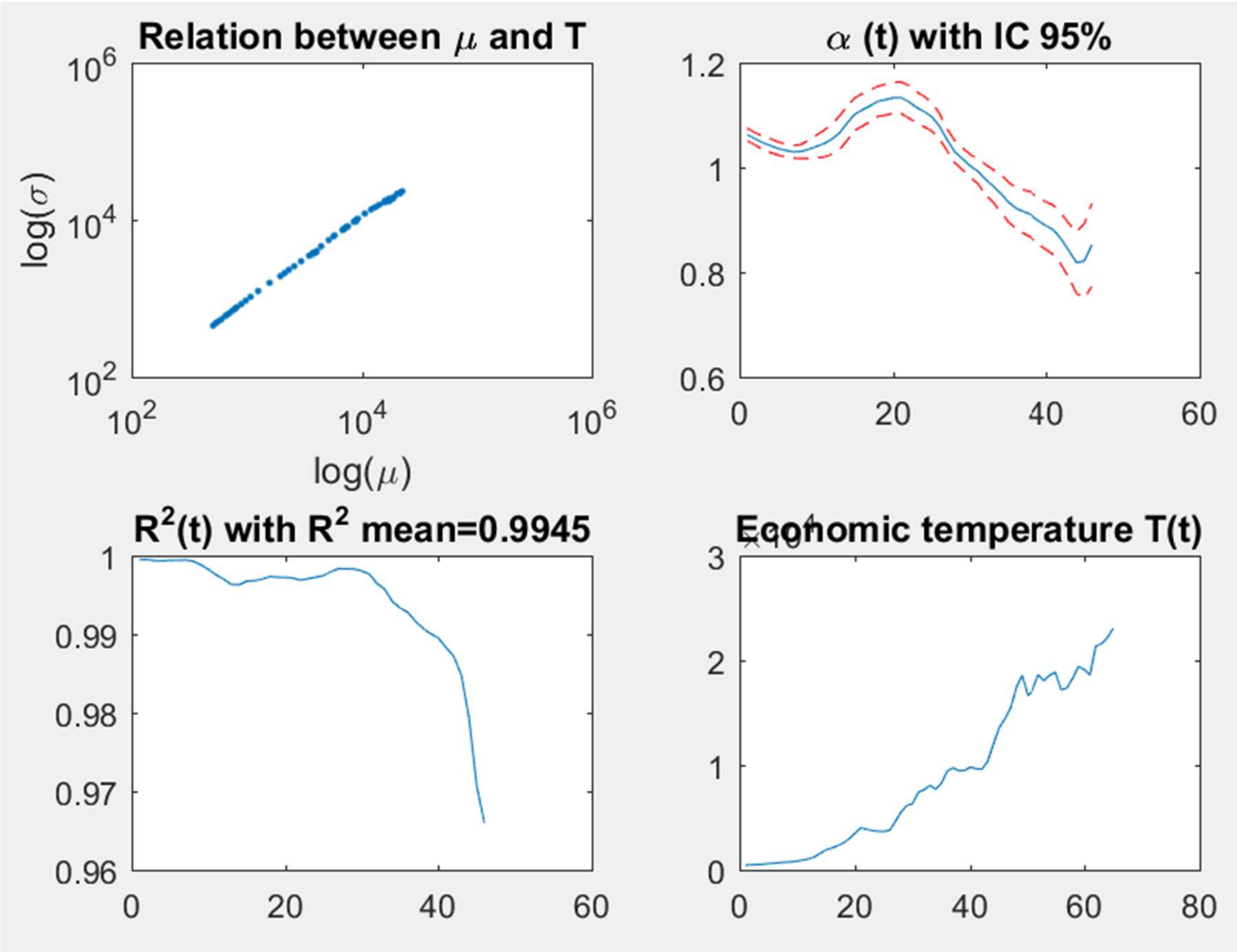
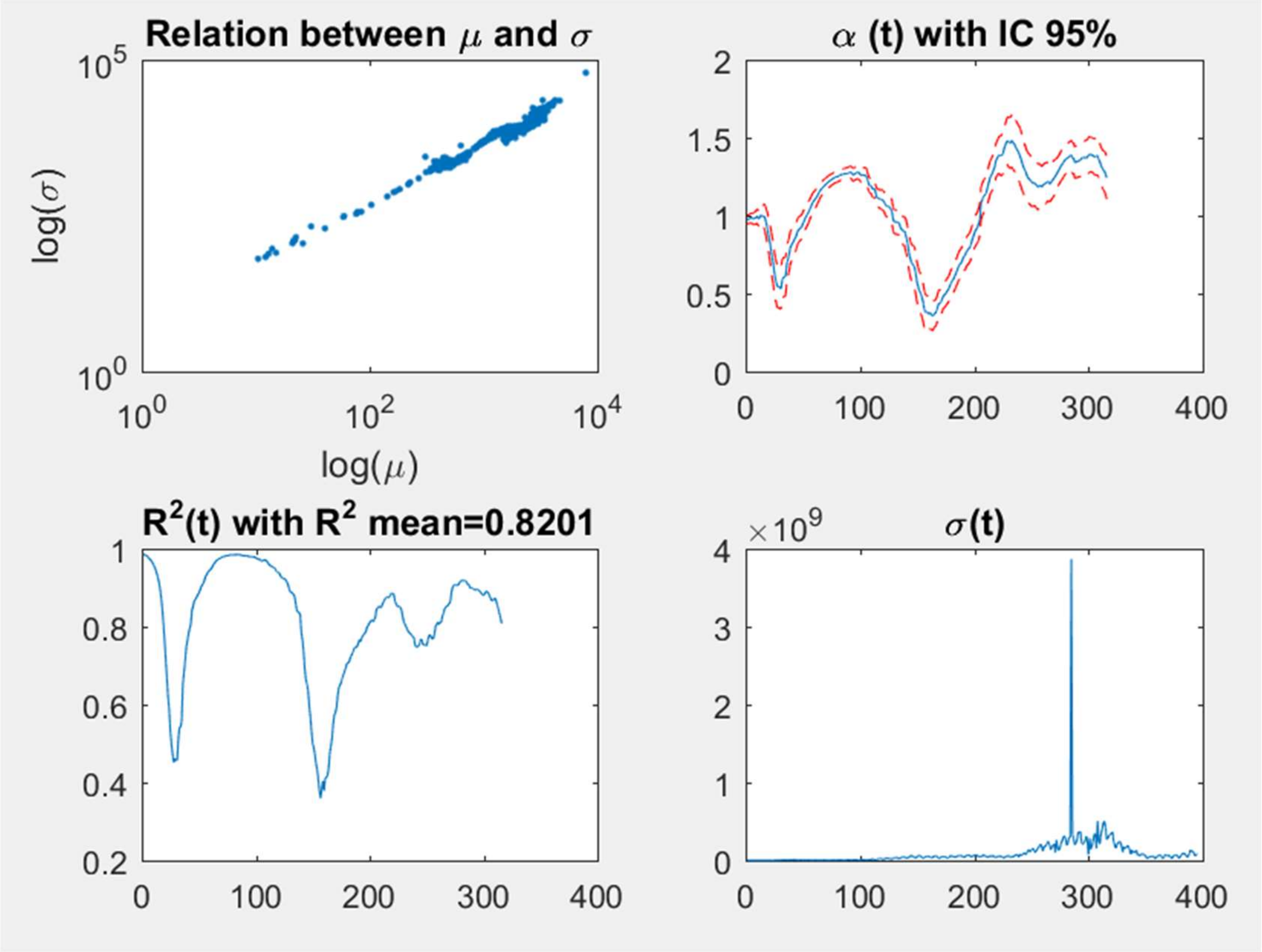
Mobile window of 20 years



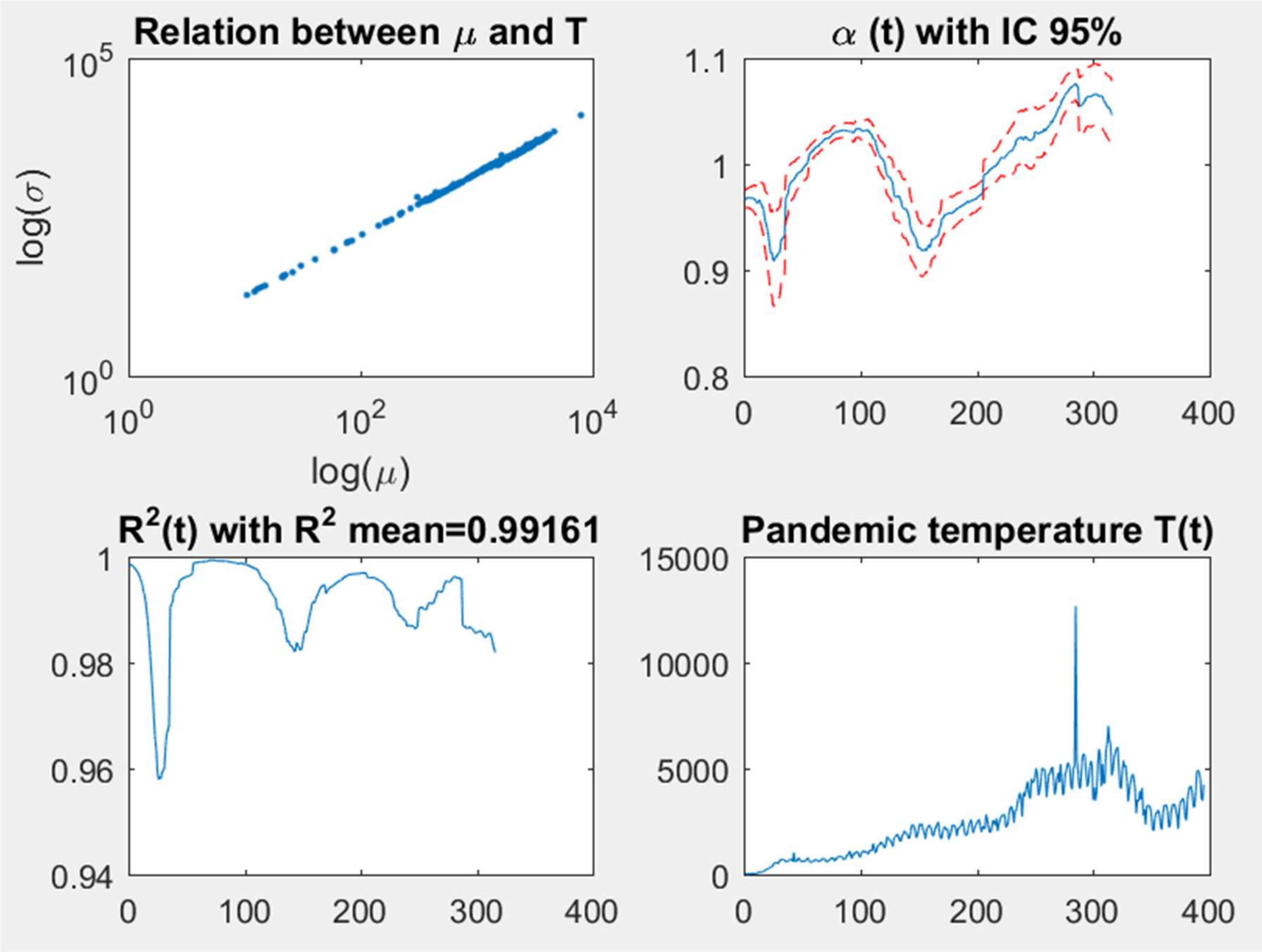
3. Time temperature scaling in different systems

Daily COVID-19 infection cases in the world for 189 countries (2020/03/01 – 2021-03-30) = 395 days –Mobile window of 80 days



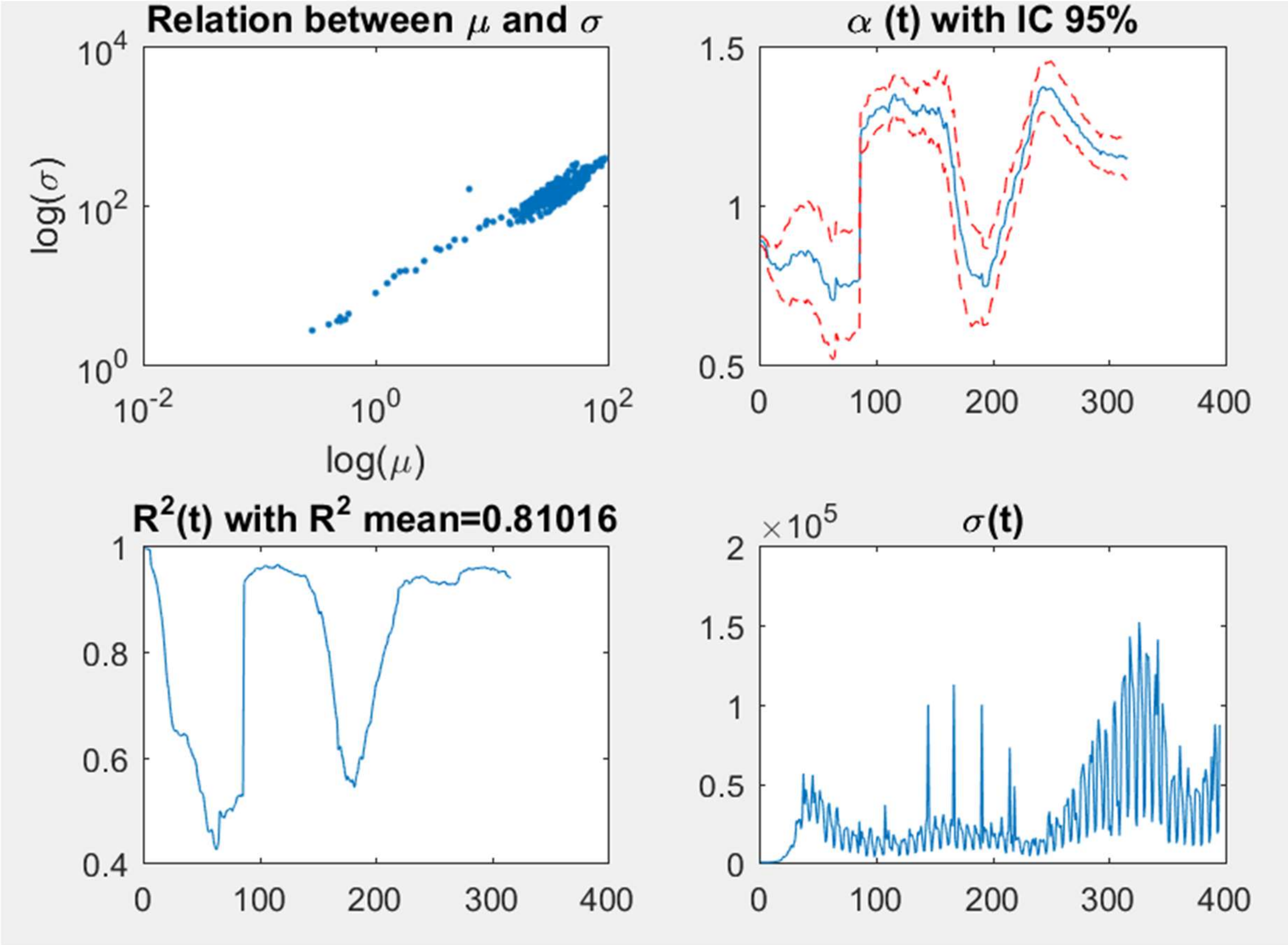
3. Time temperature scaling in different systems

Daily COVID-19 infection cases in the world for 189 countries (2020/03/01 – 2021-03-30) = 395 days –Mobile window of 80 days



3. Time temperature scaling in different systems

Daily COVID-19 deaths in the world for 189 countries (2023/03/01 – 2021-03-30) = 395 days –Mobile window of 80 days



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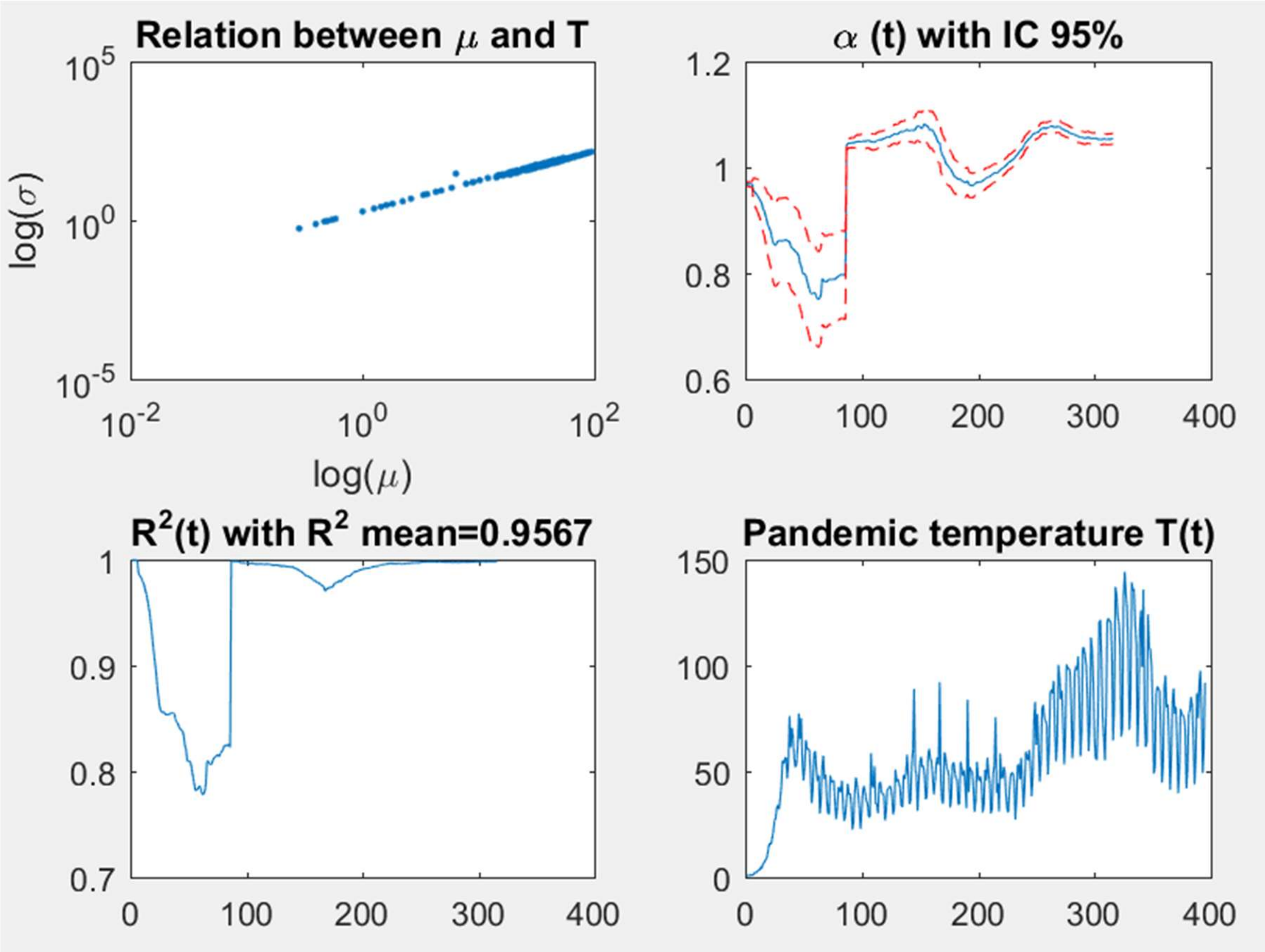
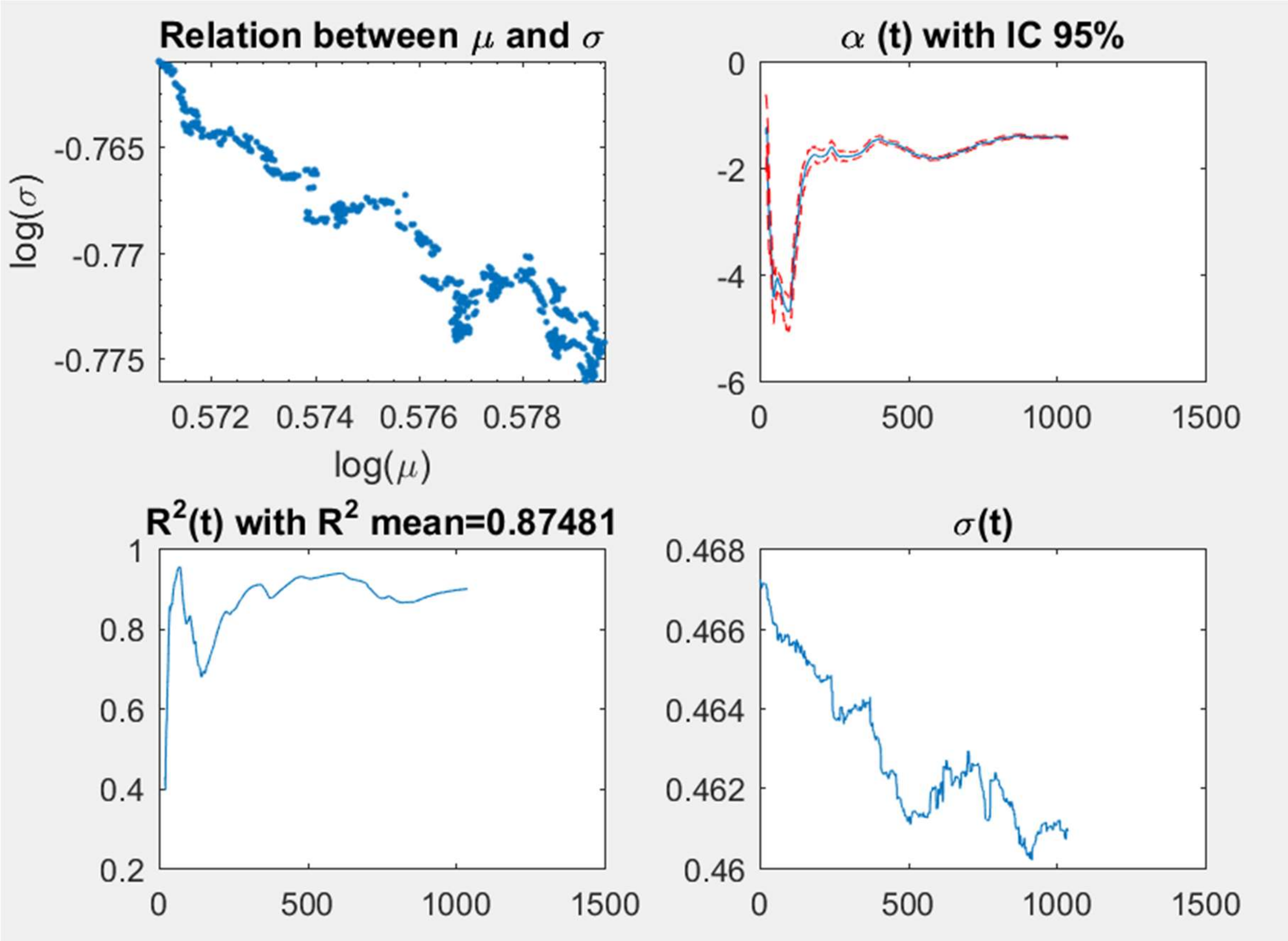


3. Time temperature scaling in different systems

Magnitud of earthquakes in Colombia (2024/01/01 – 2024-04-22) = 10008 earthquakes – Mobile window of 8968 data



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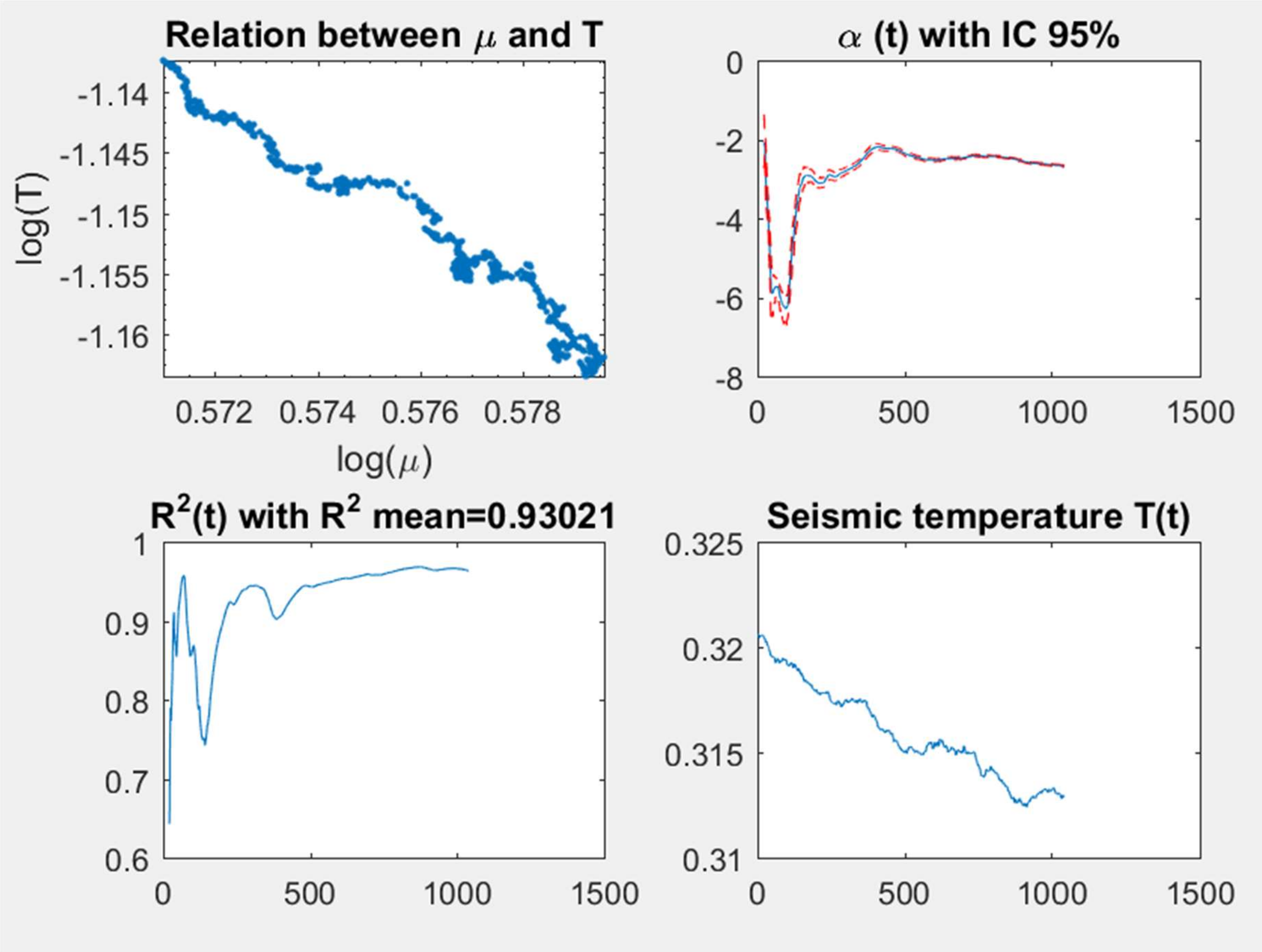
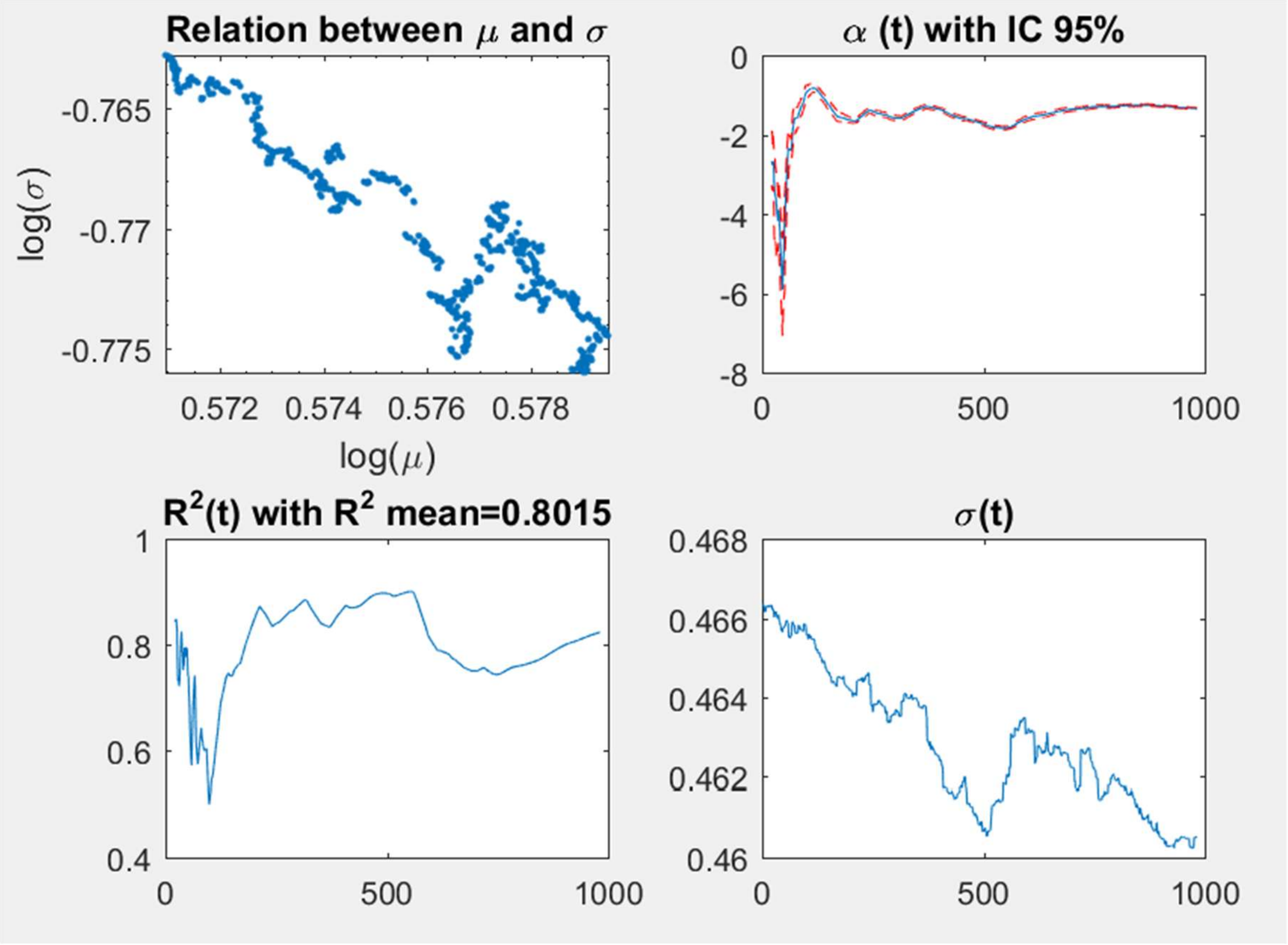


3. Time temperature scaling in different systems

Deep of earthquakes in Colombia (2024/01/01 – 2024-04-22) = 10006 earthquakes – Mobile window of 9025 data



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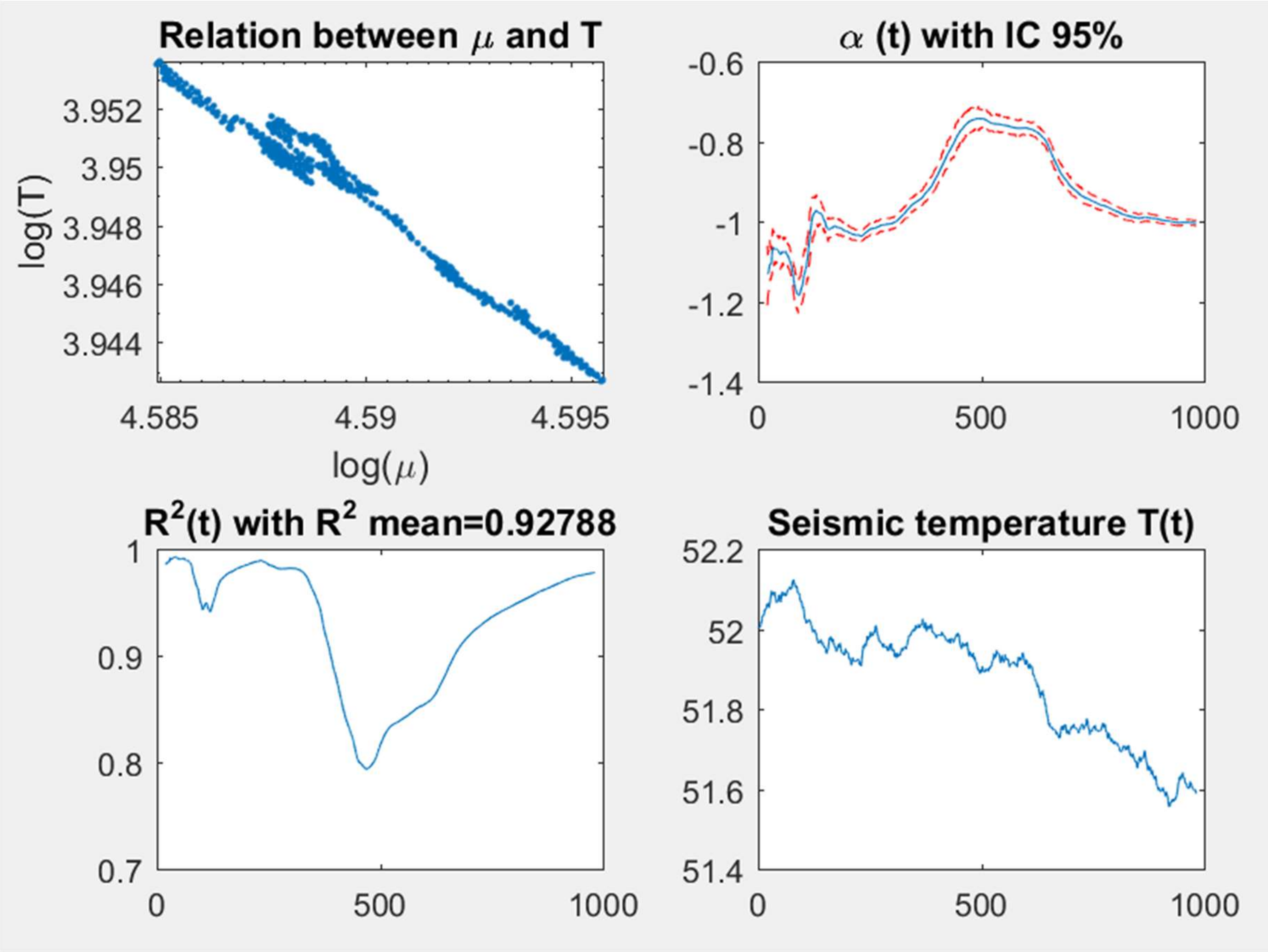

4. Conclusions

Time temperature scaling is a new property of the complex systems.

Complex systems studied through non-stationary time series clearly exhibit the property of time temperature scaling.

For the time series studied, in all cases the time temperature scaling is characterized by power laws that present better qualities of fit with respect to those obtained in the time fluctuation scaling.

In the above sense, it is possible to affirm that the time temperature scaling is a property that is more robust in the complex systems studied with respect to the time fluctuation scaling.

A task to be carried out is to be able to describe the time temperature scaling using the path integral formalism.

Thanks