Scaling limits of random curves via Schramm Loewner Evolution

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5th Workshop on Statistical Physics

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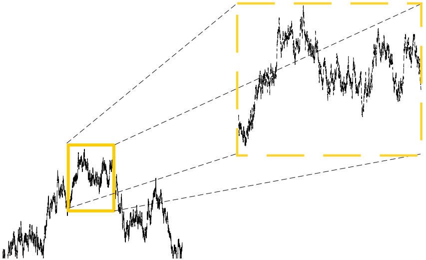
Motivation

- At the critical point, many two-dimensional systems exhibit three universal properties:
  1. *Scale invariance* expresses self-similarity and independence from microscopic details.



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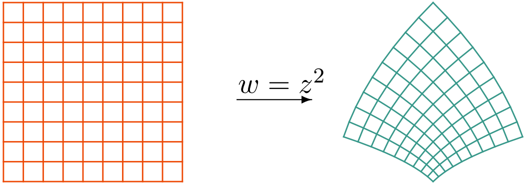

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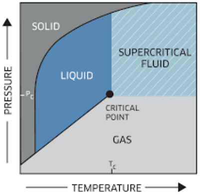
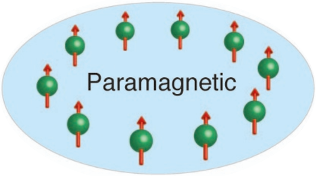
  1. *Scale invariance* expresses self-similarity and independence from microscopic details.
  2. *Conformal invariance* preserves local angles and combines rotations, translations, and dilations.
  3. *Universality* classifies systems of different origins sharing the same critical exponents.





Foundations of Schramm Loewner Evolution

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- In 1923, Karl Loewner showed that the growth of a curve can be described using conformal mappings:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}, \quad \text{with} \quad g_0(z) = z, \tag{1}$$

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- In 1999, Oded Schramm proposed that ξ_t be a Brownian motion:

$$\xi_t = \sqrt{\kappa} B_t, \tag{2}$$

  where $\kappa \in [0, \infty)$ acts as a diffusion coefficient. The choice of ξ_t gave rise to the *Stochastic (Schramm) Loewner Evolution - SLE.*

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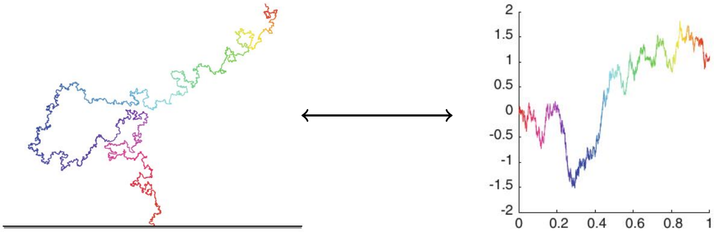

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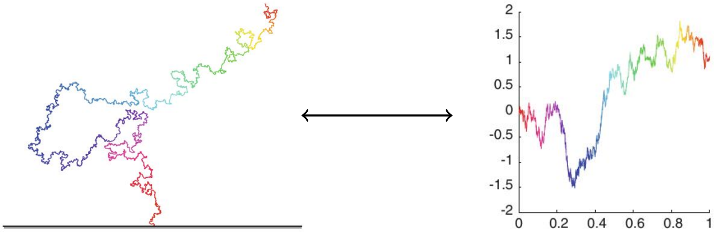
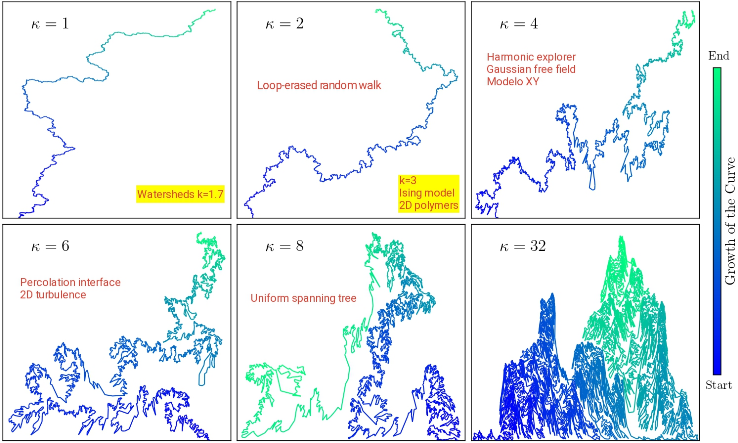


κ

Phases of SLE

Dilute phase ($0 < \kappa \leq 4$), dense phase ($4 < \kappa < 8$), space-filling phase ($\kappa \geq 8$).



Numerical methods of SLE

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To verify the compatibility of a curve with SLE, κ must be computed:

1. Fractal dimension: $N_\epsilon \sim \epsilon^{-d_f}$.

For a SLE curve:

$$d_f = \begin{cases} 1 + \frac{\kappa}{8} & \kappa < 8 \\ 2 & \kappa \geq 8 \end{cases} \tag{3}$$

For the outer edge: $d_{f_{\text{acc}}} = 1 + \frac{2}{\kappa}$

The conformal invariance is verified:

$$(d_f - 1)(d_{f_{\text{acc}}} - 1) = \frac{1}{4}$$

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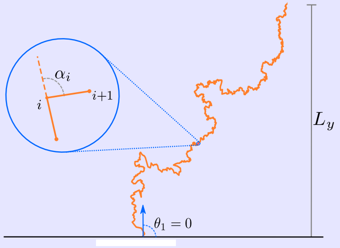
2. Winding angle: $\theta_{i+1} = \theta_i + \alpha_i$



$$\langle \theta^2 \rangle = b + \frac{\kappa}{4} \ln(L_y)$$

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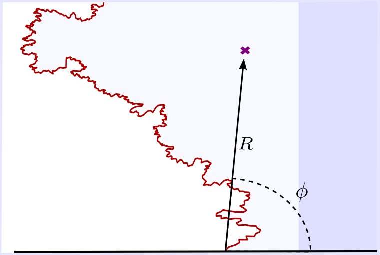
3. The left passage probability The probability that the curve passes to the left of a given point.



$$Q(\kappa) = \frac{1}{N} \sum_R \sum_\phi \left[P(\phi, R) - P_\kappa(\phi)\right]^2$$

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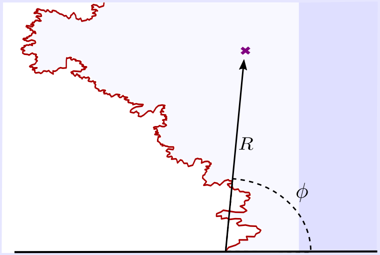

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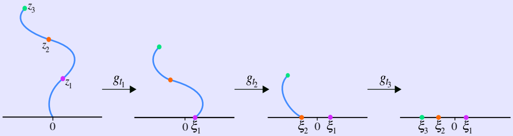
4. Direct SLE: parametrizing the curve,



- $\delta t_j = t_j - t_{j-1} = (Im\{z_j\})^2/4$
- $\xi_{t_j} = Re\{z_j\}$

$$\langle \xi_t^2 \rangle = \kappa t.$$

Coastlines are compatible with SLE?

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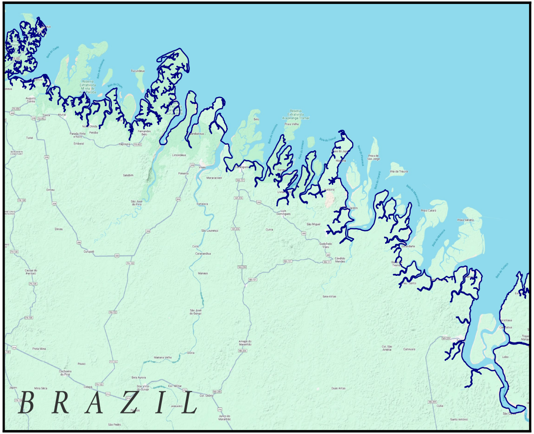


Figure 1: Coast of Brazil

- Mandelbrot[a] established coastlines are fractal.
- Fractal dimensions were calculated for the coastlines of Australia, China, the United States, Great Britain, and others.
- Coastlines are topographic lines derived from the Earth's surface, a self-affine object (anisotropic rescaling).

[a] How Long Is the Coast of Britain? Mandelbrot. Science (1967)

Random surfaces using FFM

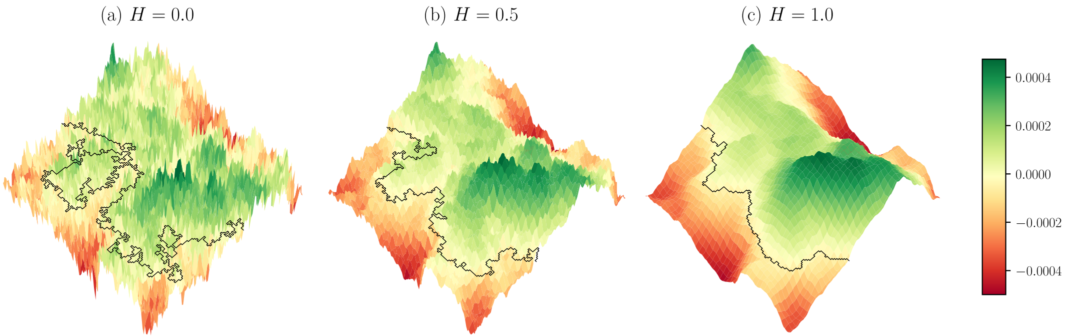


Figure 2: Surfaces with different Hurst exponents. At $h = 0$, an isoheight contour appears, resembling a coastline.

Percolation in random surfaces

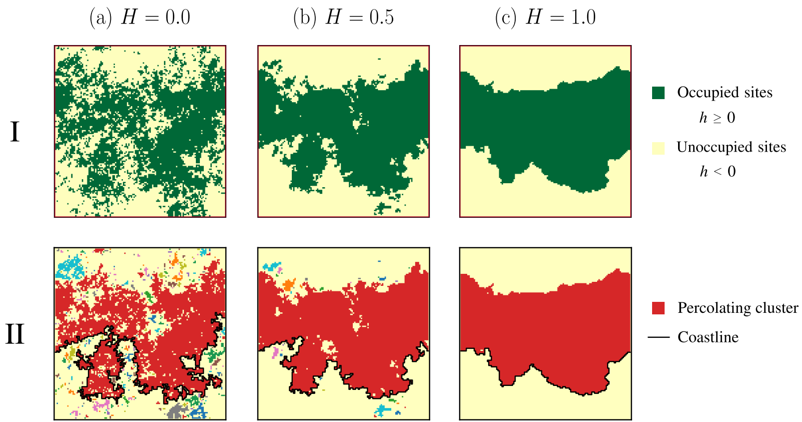
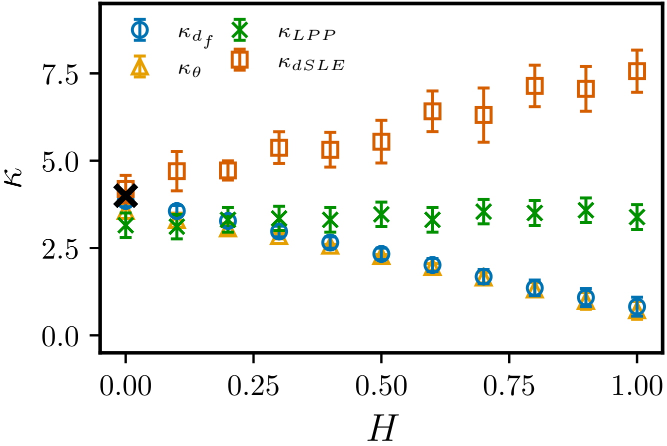


Figure 3: Extraction of a coastline using percolation.

These lines are studied with SLE.

Diffusion parameter for coastlines



Only for $H = 0$ (Gaussian Free Field) are contour lines described by SLE.

Invasion Percolation (IP) trees

Viscous fluid displacement in a porous medium by injection.

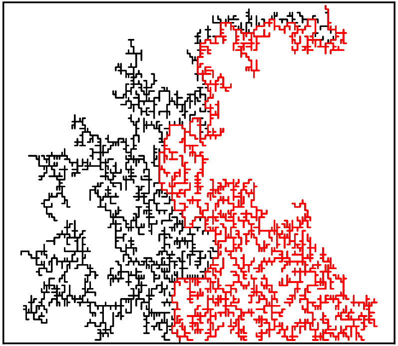


Figure 4: NTIP tree traversed using depth-first search (DFS).

- Oil and gas extraction from reservoirs, groundwater hydrology, contaminant transport in soils, etc.
- Non trapping IP is believed to belong to the same universality class as random percolation, with $d_f = 1.89$.
- Trees are traversed[a] using the Invasion order, DFS and Breadth-first search.

[a] *Loewner evolution for critical invasion percolation tree*, Abril, L. M. *et al.*, EPJ-ST (2025)

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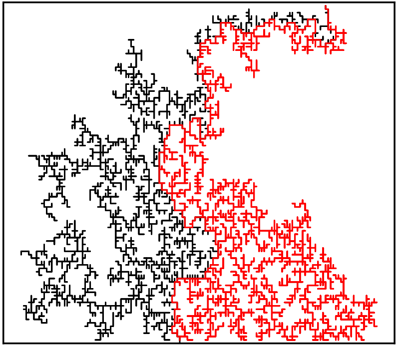

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Extensions to SLE theory must be developed to incorporate trees.

Conclusions

1. SLE characterizes 2D fractal interfaces in the scaling limit.
2. Coastlines and invasion percolation trees show inconsistencies with standard SLE.
3. These results highlight the need for extensions of SLE to account for the observed behaviour, such as fractional SLE or SLE(κ, ρ).

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References

- Abril, L. M. et al., *Coastlines violate the Schramm–Loewner Evolution*, Physica A (2024).
- Abril, L. M. et al., *Loewner evolution for critical invasion percolation tree*, EPJ ST (2025).

Thank you!