Very persistent random walkers reveal transitions in landscape topology

ICTP SAIFR

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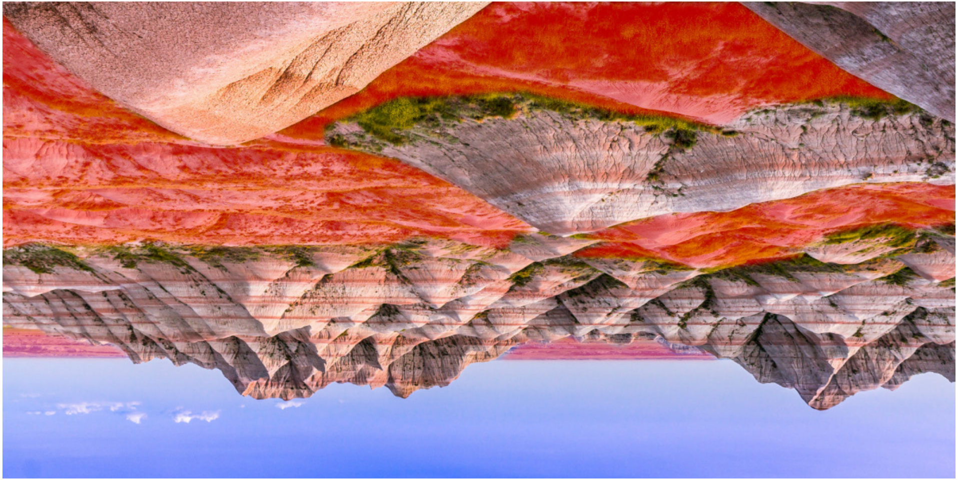

Dynamics and optimization in complex landscapes

Dynamic thresholds

Long clear that proliferation of metastable minima impedes descent

But, many metastable minima is not a sharp predictor: many methods pass some proliferations of minima but not others

Recent progress on tightly bounding performance results from *level set topology*

Example: the *overlap gap property* predicts energy density E_{alg} bounding polynomial-time optimization from below by existence of system-spanning component

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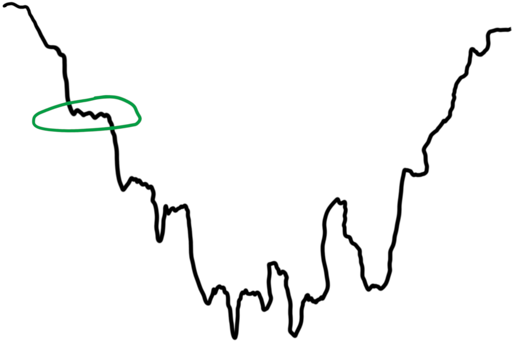

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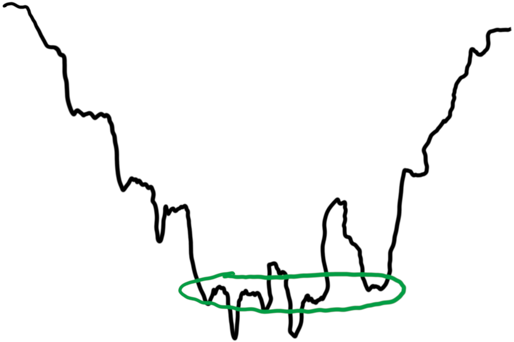

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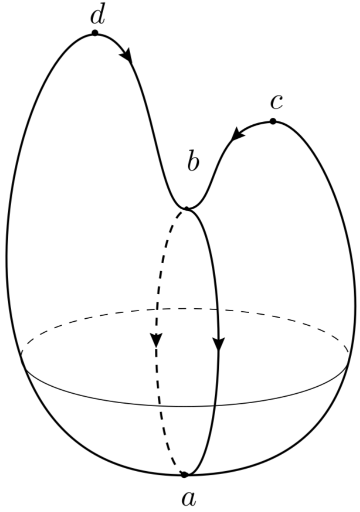

Fig. 2.13 The other sphere

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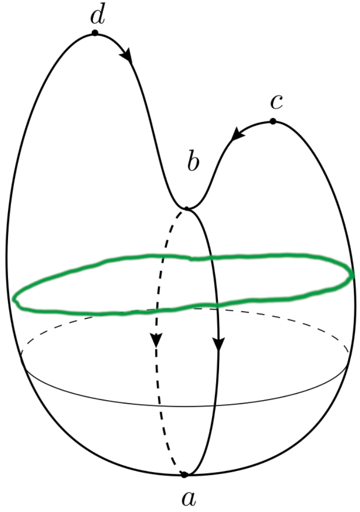

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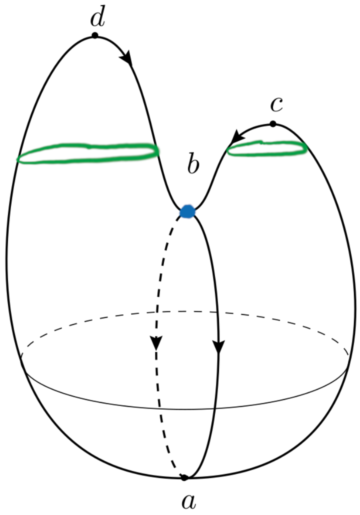

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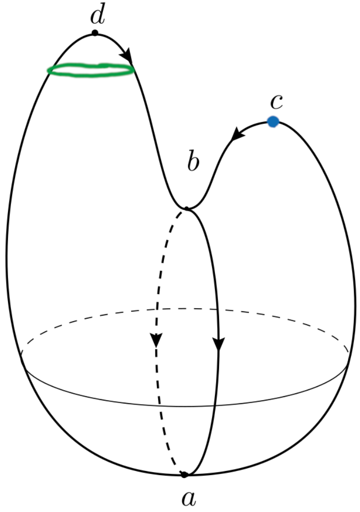

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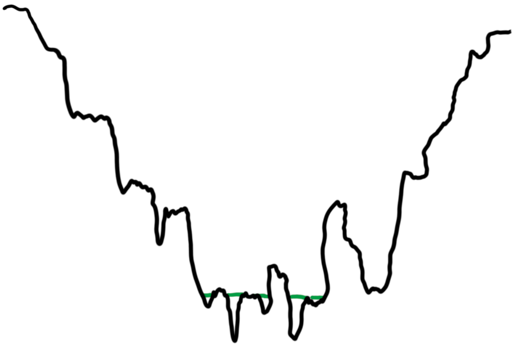


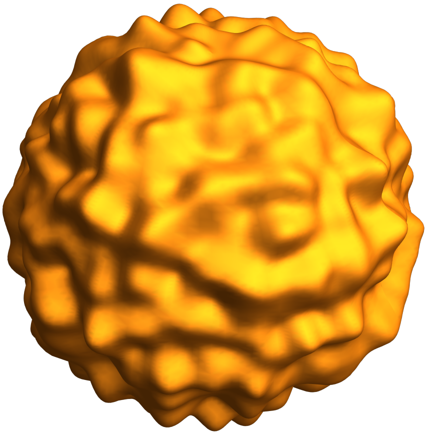
Family of models: spherical spin glasses

Random polynomials in $\boldsymbol{x} \in \mathbb{R}^N$ on sphere $\|\boldsymbol{x}\|^2 = N$ with iid Gaussian coefficients J:

$$H_p(\boldsymbol{x}) = \sum_{i_1 \cdots i_p}^{N} J^{(p)}_{i_1,\dots,i_p} x_{i_1} \cdots x_{i_p}$$

$$H(\boldsymbol{x}) = \sum_{p=0}^{\infty} a_p H_p(\boldsymbol{x})$$

$$\underbrace{H(\boldsymbol{x}) = H_p(\boldsymbol{x})}_{\text{pure } p\text{-spin}} \qquad \underbrace{H(\boldsymbol{x}) = \frac{1}{2} H_p(\boldsymbol{x}) + \frac{1}{2} H_s(\boldsymbol{x})}_{\text{mixed } p+s\text{-spin}}$$



$N = 3,\ p = 128$

Family of models: spherical spin glasses

Pure p-spin models

One variety of stationary point at each energy density; *all* marginal minima at same energy density E_{th}

"Threshold" energy E_{th} governs all dynamics: $E_{\text{th}} = E_{\text{gd}} = E_{\text{alg}}$, result of quenching, GAMP, etc

Dynamics stalls when connected component breaks up

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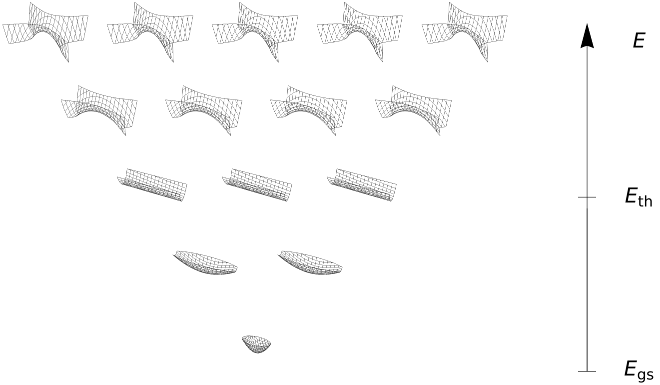

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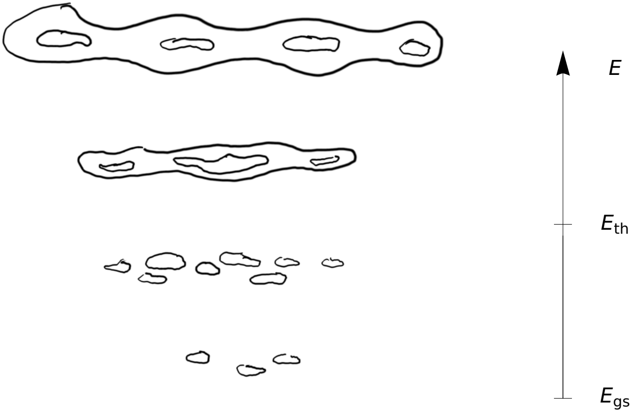

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Mixed $p+s$-spin models

Mixture of different stationary points at each energy density

E_{th} dividing mostly saddles from mostly minima no longer sharp

Breakup of spanning component occurs over range of energies

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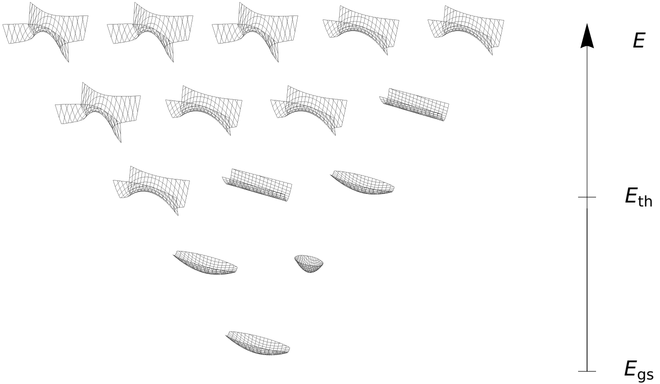

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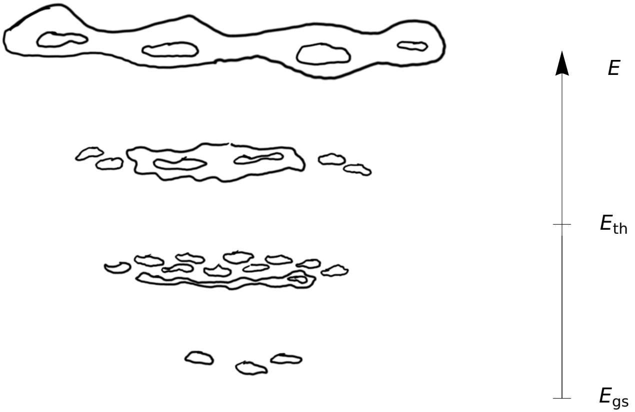

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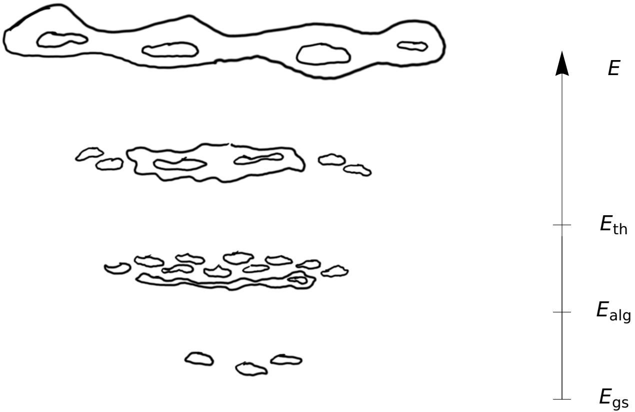

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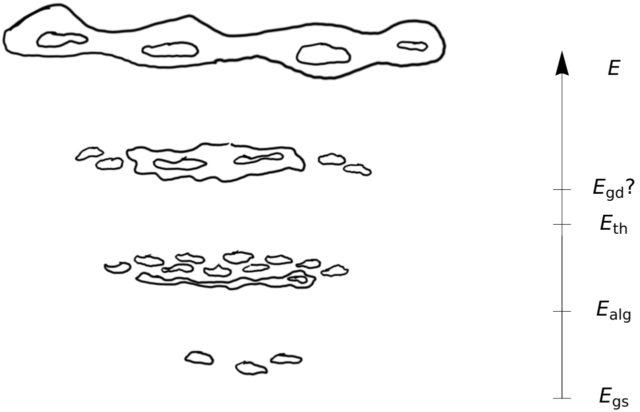

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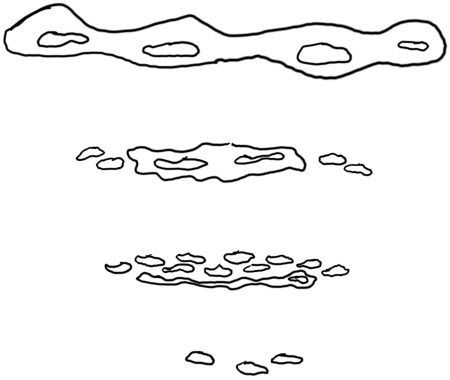
A topological determinant for worst case performance?

Idea: 'stupid' descent will always belong to 'typical' piece of level set

At what energy do typical parts of the level set stop belonging to a spanning component?

Or: at which point does a random walk starting from a typical point in the level set lose ergodicity, e.g., the correlation function $C(\tau) = \frac{1}{N}\boldsymbol{x}(0) \cdot \boldsymbol{x}(\tau)$ has asymptotic

$$\lim_{\tau\to\infty} C(\tau) = q > 0?$$



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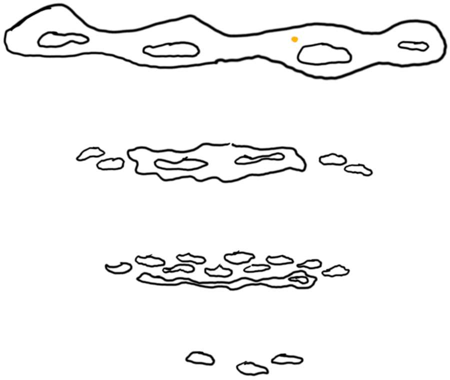

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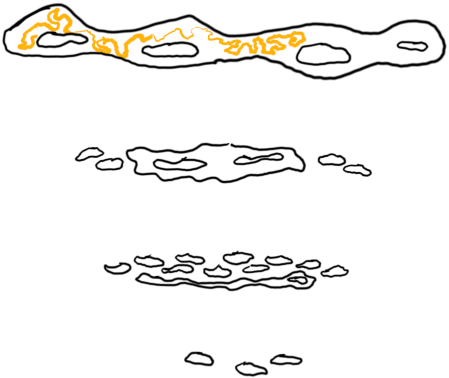

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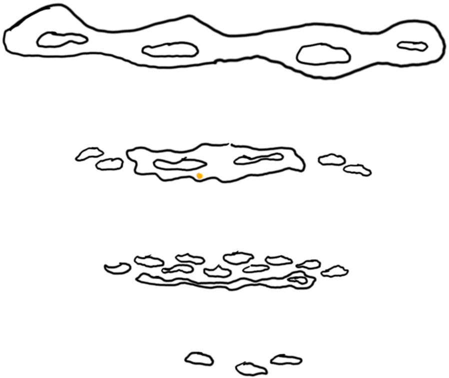

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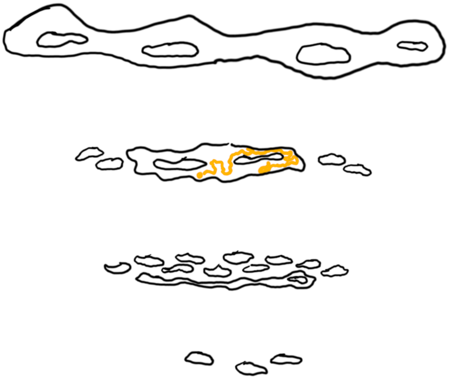

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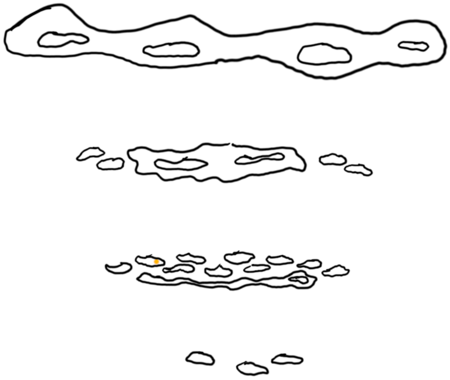

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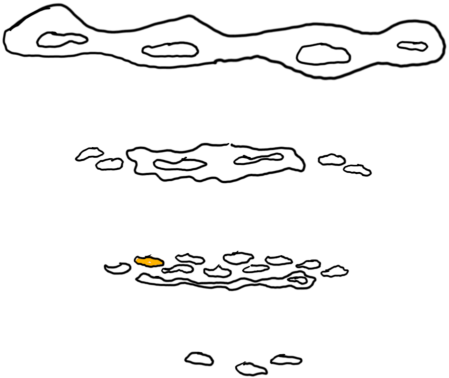

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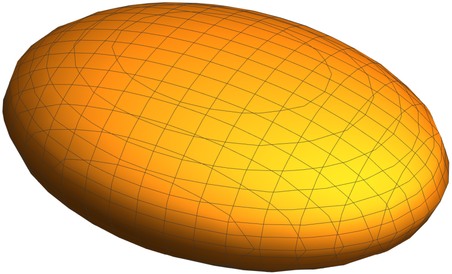
Ergodicity of passive walks

When the walk is Brownian, behavior equivalent to equilibrium dynamics

In equilibrium, ergodicity breaking occurs at temperature T_d corresponding to energy density E_d

E_d is not a topologically relevant energy! Caused by *entropic* barriers

Example: 2-spin model has convex landscape with $E_\mathrm{gs} = E_\mathrm{th} = -1$, but $E_\mathrm{d} = -\frac{1}{2}$



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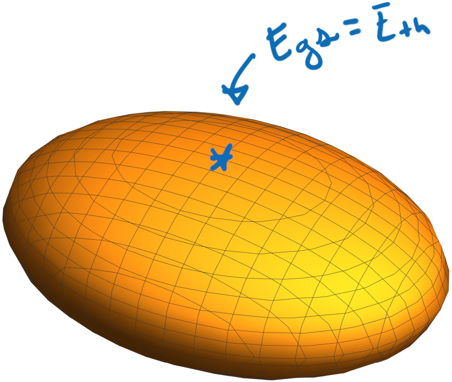

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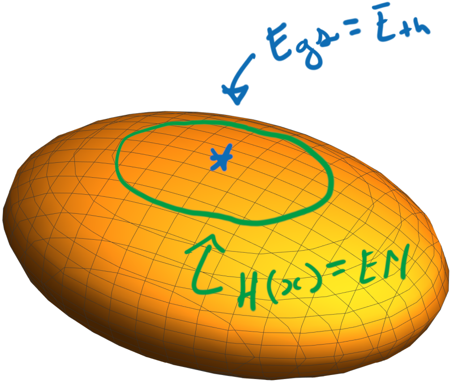

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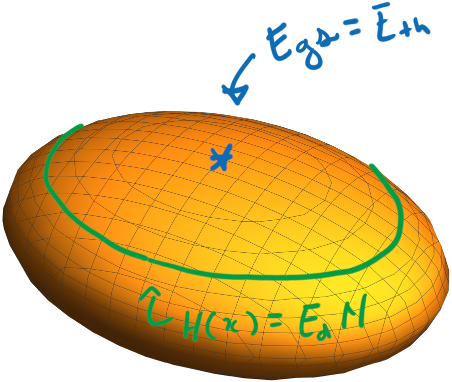

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Ergodicity of active walks

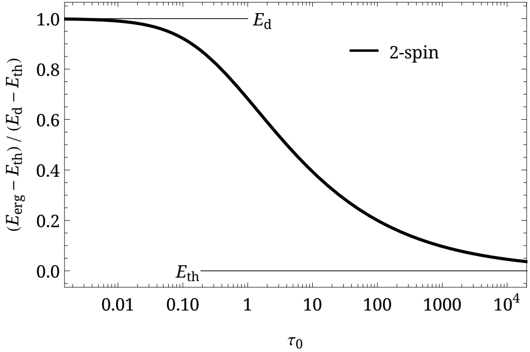
Exact results

Consider instead walks driven by noise with persistence time τ_0, or with covariance

$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{1}{\tau_0} e^{-|t-t'|/\tau_0}$$

Exactly solvable in the 2-spin model, which is convex *not* complex

Increasing τ_0 drives ergodicity breaking transition E_{erg} from $E_{\mathrm{d}} = -\frac{1}{2}$ to $E_{\mathrm{th}} = E_{\mathrm{gs}} = -1$



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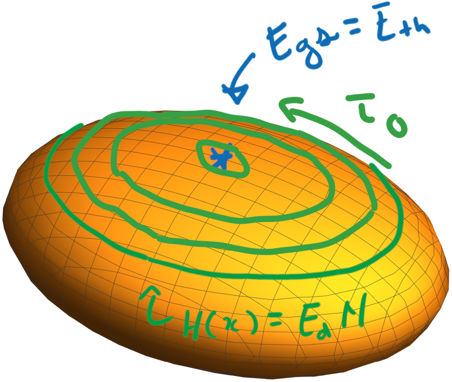

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Exact solutions not possible in other models

Equations discretized and solved numerically by iteration to and from frequency domain

In pure p-spin models, ergodicity-breaking transition E_{erg} interpolates between equilibrium transition E_{d} and threshold E_{th}

Strong persistence beats entropic barriers

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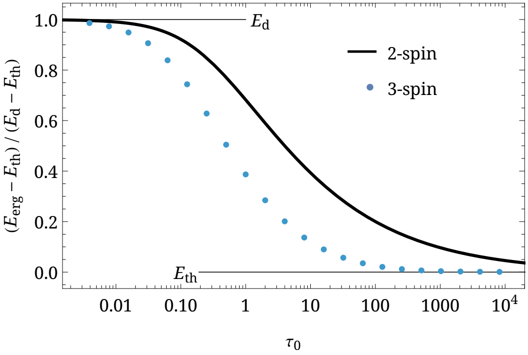

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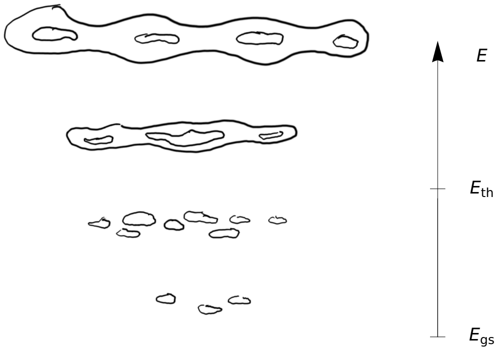

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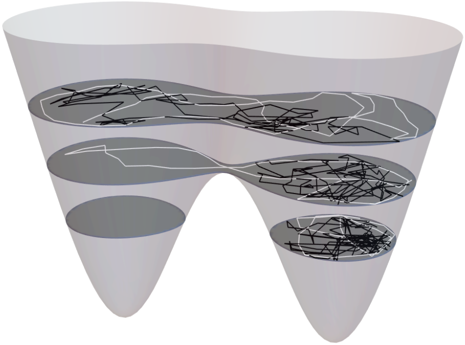

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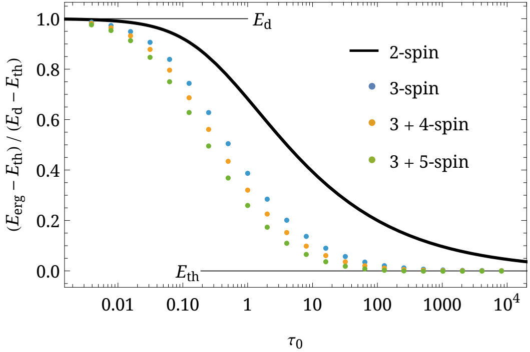

Ergodicity of active walks

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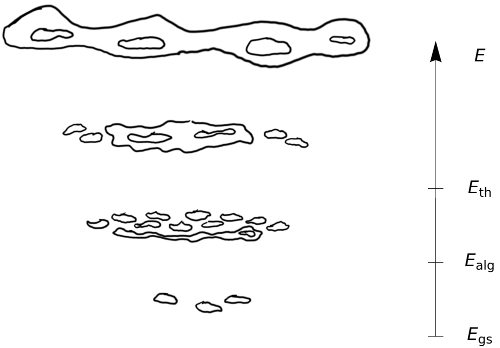

What about gradient descent dynamics?

Is $E_{\text{gd}} = E_{\text{th}}$ in mixed spherical models?

Old results suggest no. . .

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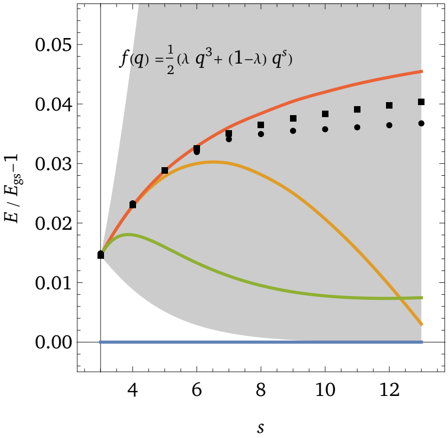

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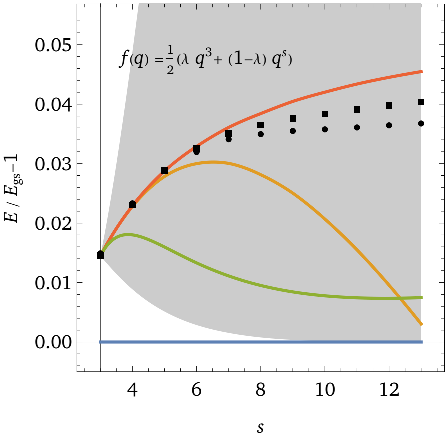

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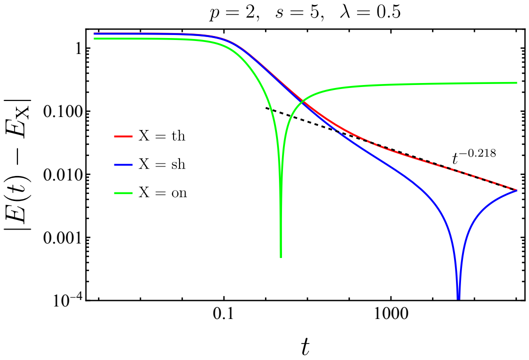

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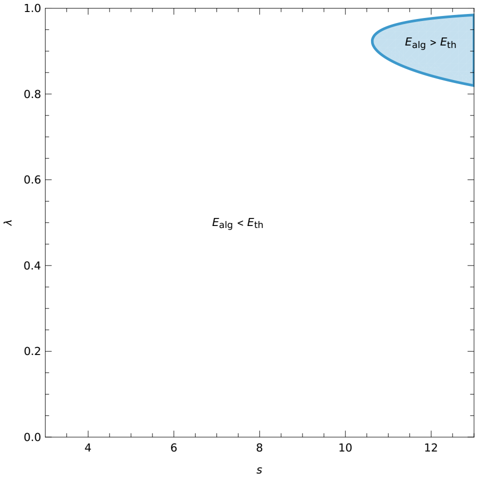

Is this relevant outside mean field?

Random Lorentz gas is a simple model of glass formation: spherical obstacles randomly fill space, configuration is single point tracer

Seeming inconsistency between density $\hat{\varphi}$ of glass transition versus that of percolation transition

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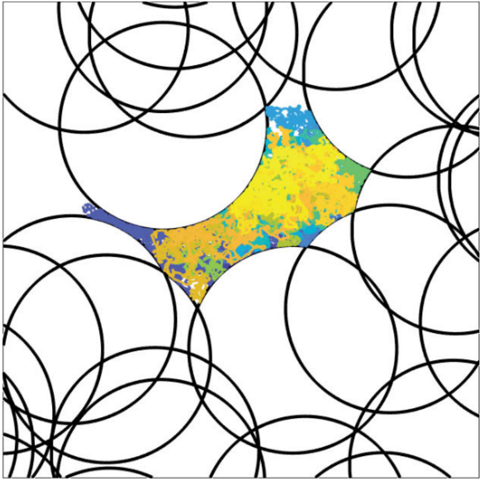

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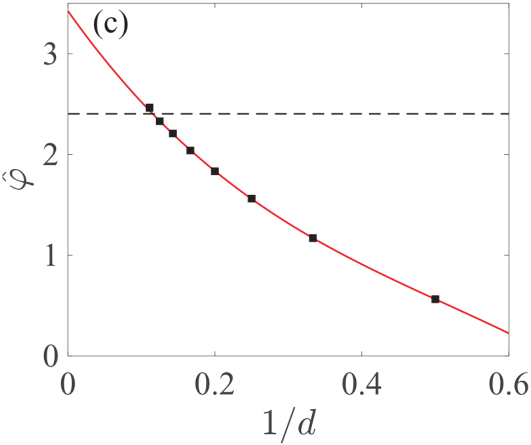

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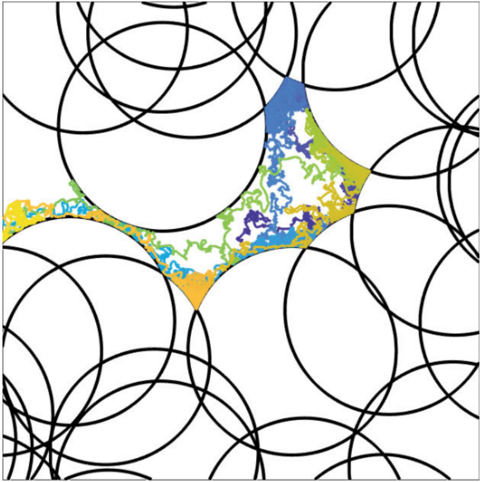

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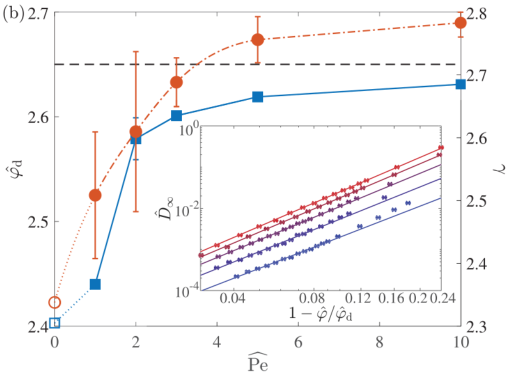
Ergodicity-breaking density $\hat{\varphi}_\mathrm{d}$ (●) as a function of Péclet number for $d = 12$ RLG compared with percolation threshold (- - -)



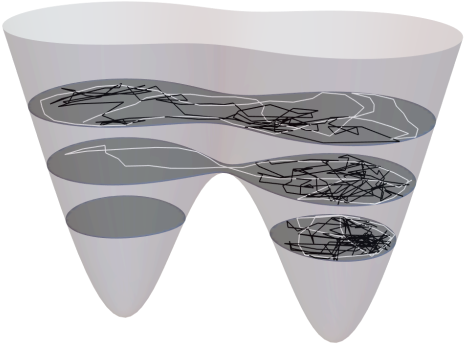
Conclusions

Topology of constant energy level sets crucial to bounding physical dynamics and algorithmic performance

Ergodicity of infinitely persistent random walks measures typicality of spanning component in level set

Spanning component becomes atypical at threshold energy in (certain) mixed spherical models, suggesting it as upper bound and destination for gradient descent

J Kent-Dobias, "Very persistent random walkers reveal transitions in landscape topology", Physical Review Letters 136, 117401 (2026)



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Bounding performance from below

Overlap $q = \frac{1}{N}\boldsymbol{x} \cdot \boldsymbol{x}'$ is scalar product between two system configurations

The overlap gap property (OGP) bounds from below the performance of generic algorithms at energy density E_{alg}

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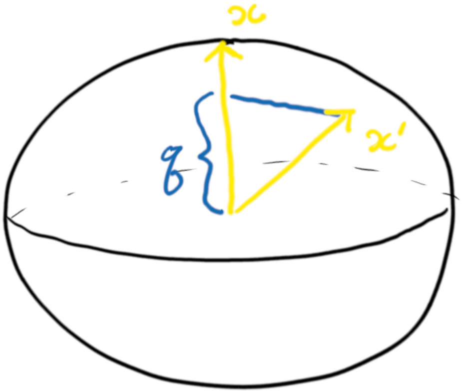
Often saturated by generalized approximate message passing algorithms (GAMP) for mean-field systems



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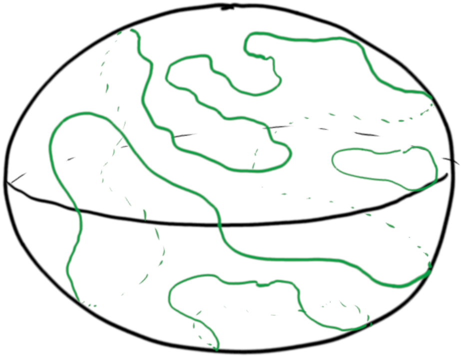

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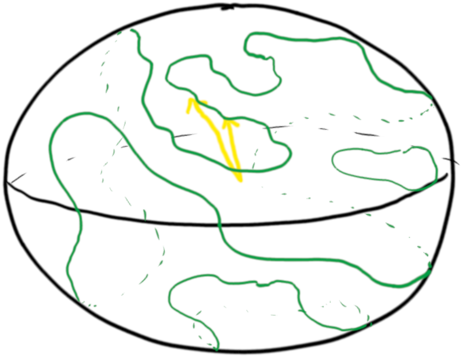

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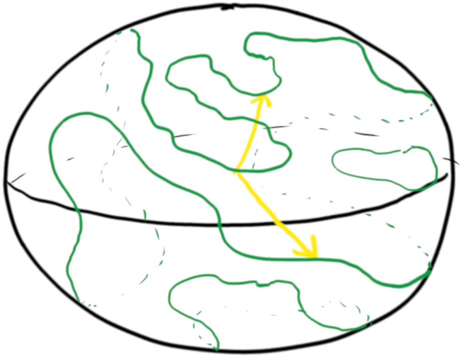

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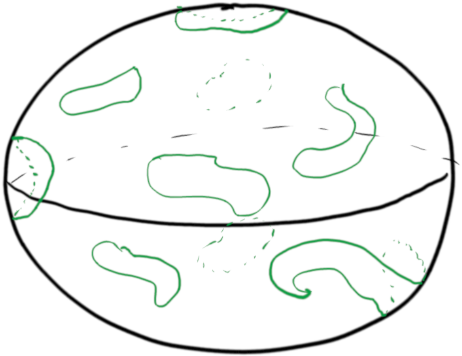

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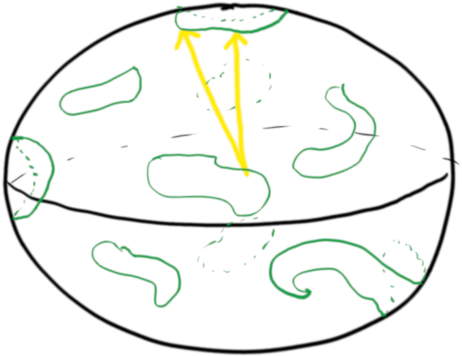

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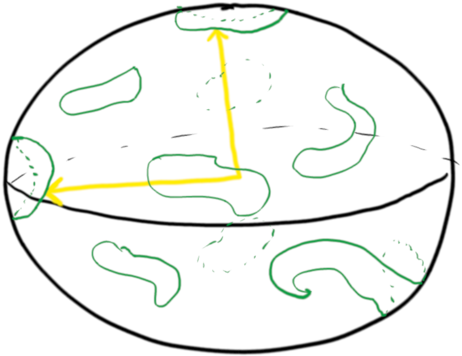

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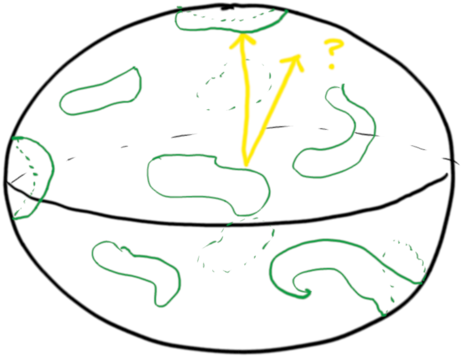

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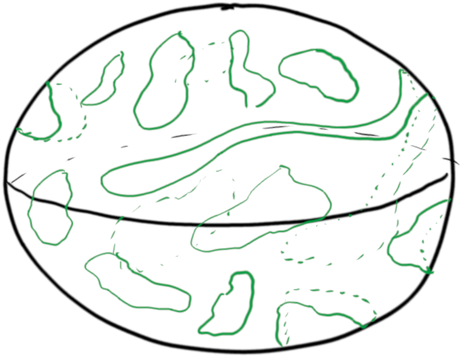

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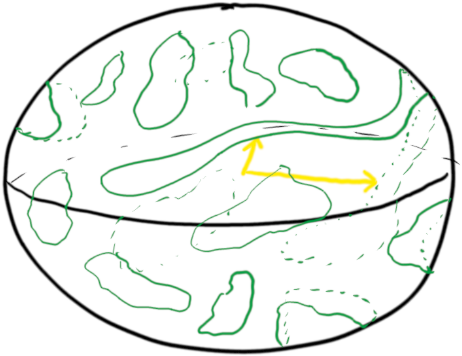


Random walks on microcanonical configuration space

Random walk with Gaussian noise $\langle \xi_i(t)\xi_j(s)\rangle = \delta_{ij}\Gamma(t-s)$ constrained by spherical configuration space

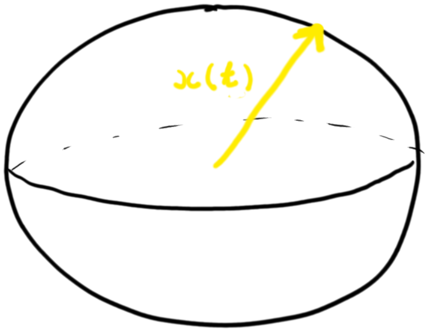
$$g_1(\boldsymbol{x}) = \frac{1}{2}(\|\boldsymbol{x}\|^2 - N)$$

and energy-level set

$$g_2(\boldsymbol{x}) = H(\boldsymbol{x}) - NE$$

using dynamic Lagrange multipliers:

$$\begin{aligned}\frac{\partial \boldsymbol{x}(t)}{\partial t} &= \boldsymbol{\xi}(t) - \mu(t)\boldsymbol{\nabla} g_1(\boldsymbol{x}(t)) - \beta(t)\boldsymbol{\nabla} g_2(\boldsymbol{x}(t)) \\ &= \boldsymbol{\xi}(t) - \mu(t)\boldsymbol{x}(t) - \beta(t)\boldsymbol{\nabla} H(\boldsymbol{x}(t))\end{aligned}$$



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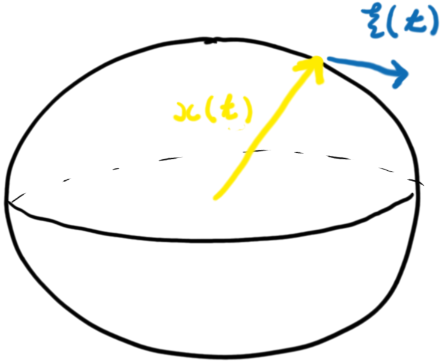

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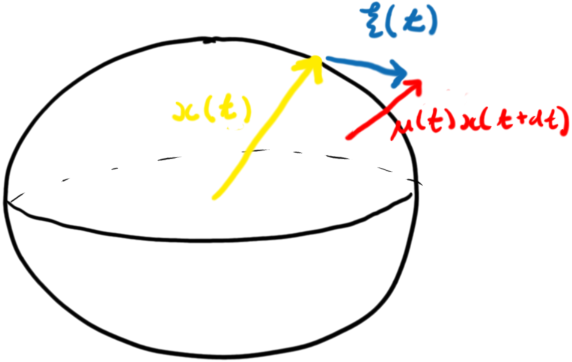

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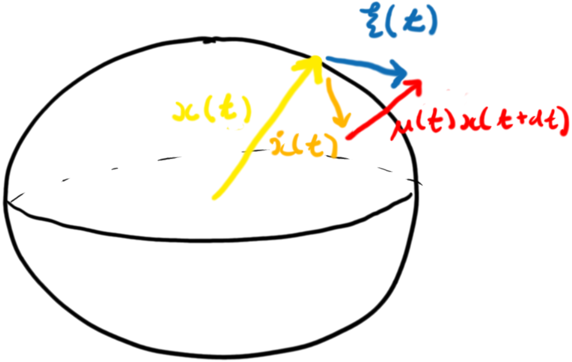

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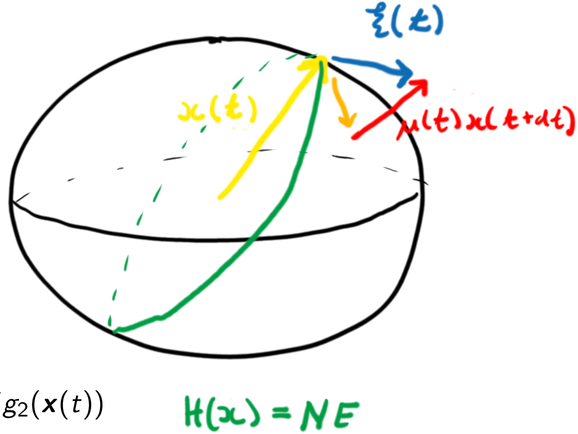

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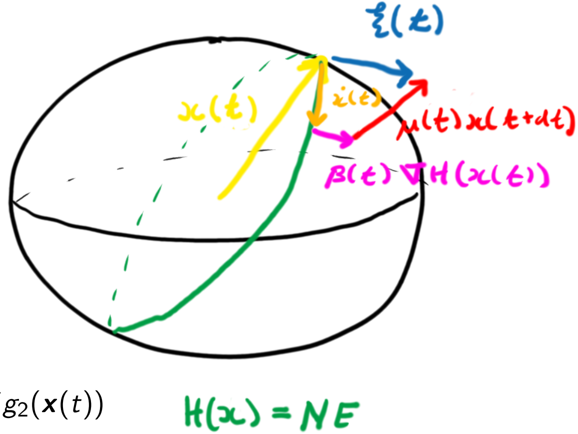

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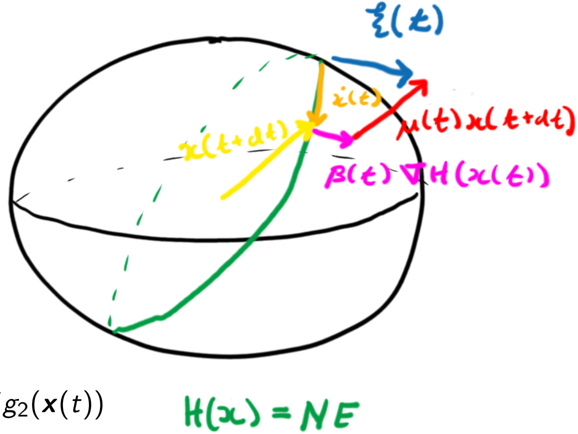


A topological determinant for worst case performance?

Hard to find very low-energy minima; but also hard to find very high-energy minima!

Can we bound typical performance from above? (e.g., gradient flow)

Marginal minima bound but not tightly

J Kent-Dobias, "Conditioning the complexity of random landscapes on marginal optima", Physical Review E 110, 064148 (2024)

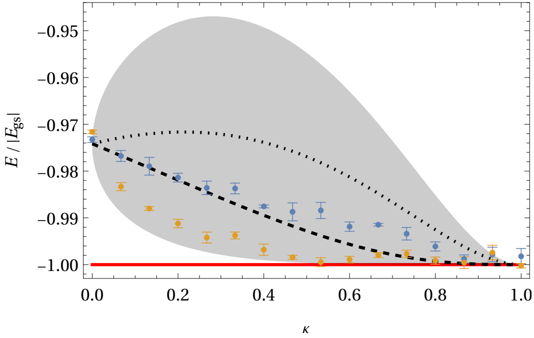
J Kent-Dobias, "Algorithm-independent bounds on complex optimization through the statistics of marginal optima", arXiv preprint, July 2024, arXiv:2407.02092 [cond-mat.dis-nn]

Transition in Euler characteristic of level sets at E_{sh} matches some imprecise numerics but mechanism unclear

J Kent-Dobias, "On the topology of solutions to random continuous constraint satisfaction problems", SciPost Physics 18, 158 (2025)

G Folena and F Zamponi, "On weak ergodicity breaking in mean-field spin glasses", SciPost Physics 15, 109 (2023)

Range of marginal minima compared with performance of gradient descent (●) and message passing (●) for a set of random landscape models parameterized by κ



A topological determinant for worst case performance?

Hard to find very low-energy minima; but also hard to find very high-energy minima!

Can we bound typical performance from above? (e.g., gradient flow)

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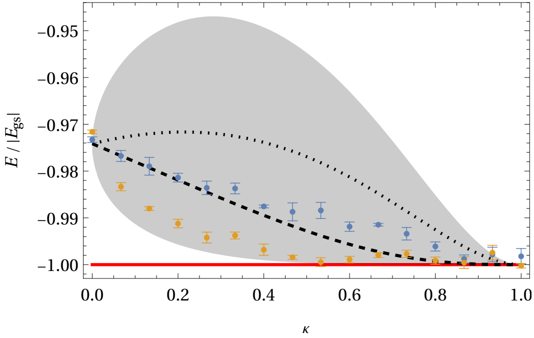

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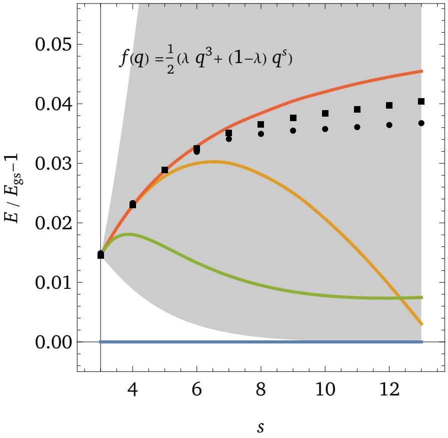
Estimated performance of gradient descent (●, ■) versus E_{sh}, E_{th}, E_{alg}, and E_{gs} for 3+s-spin spherical spin glasses



DMFT for spherical spin glasses

Path-integral methods yield closed equations for correlation and response functions under assumption of time-translation invariance:

$$\left(\frac{\partial}{\partial\tau}+\mu\right)C(\tau)=2\int d\sigma\,\Gamma(\tau-\sigma)R(-\sigma)+\beta^2\int d\sigma\,R(\tau-\sigma)f''\big(C(\tau-\sigma)\big)C(\sigma)$$
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where $\quad f(q)=\frac{1}{2}\sum_{p=0}^{\infty}a_p^2q^p \quad$ if $\quad H(\boldsymbol{x})=\sum_{p=0}^{\infty}a_pH_p(\boldsymbol{x})$

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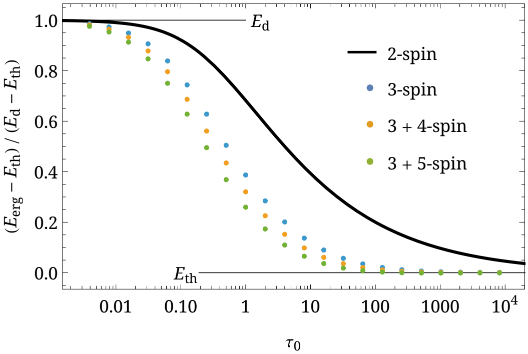
Ergodicity of active walks

Numeric results

- Ergodicity-breaking transition in mixed models also appears to approach threshold at infinite persistence time τ_0
- Difficult to conclude with power-law fits: difference in near-threshold energies much smaller than $E_d - E_{th}$
- Scaling of Lagrange multipliers near the transition suggests E_{th} is the asymptotic transition energy



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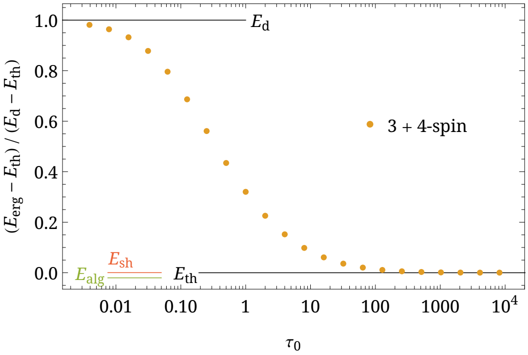

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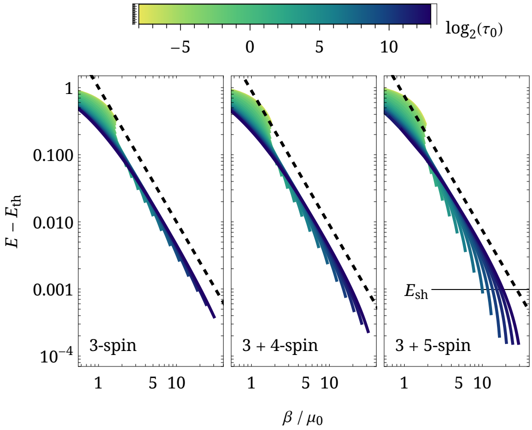

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