

Two-Point Correlation in Dyson Gases Out-of-Equilibrium

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- 1 Coulomb gas as a physical model

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- ② Mathematical Framework

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 - 3.1 $\beta = 2$: The N free-Fermion problem
 - 3.2 Initial SetUp
 - 3.3 Ground state and stationary probability density

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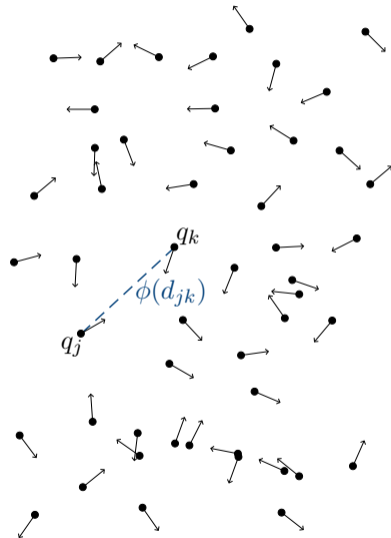
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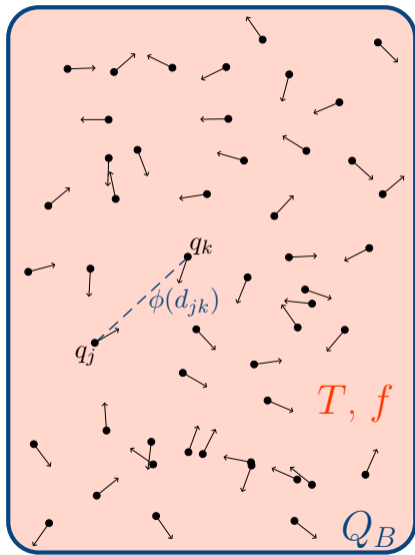
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- ⑦ $N > 2$ Case

From 2D-Poisson equation:

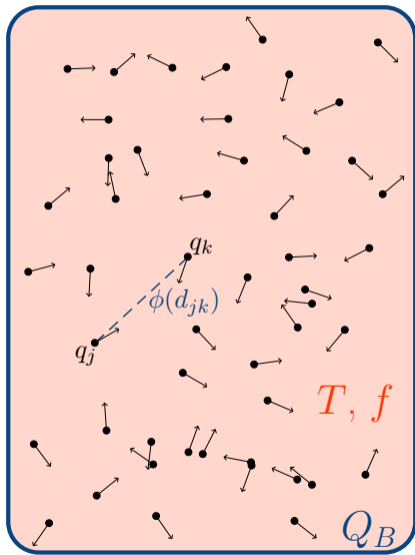
$$\phi(\mathbf{r}_k, \mathbf{r}_j) \sim -\ln(\|\mathbf{r}_k - \mathbf{r}_j\|)$$





- The effective inverse temperature:

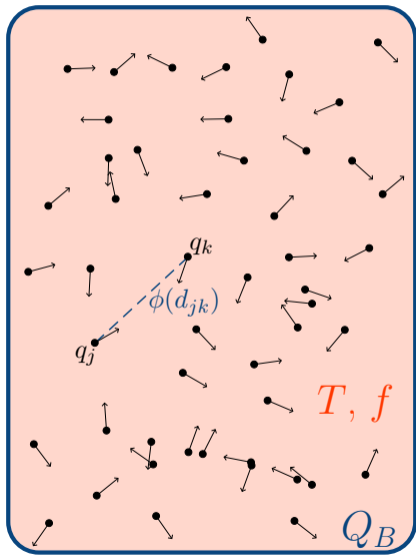
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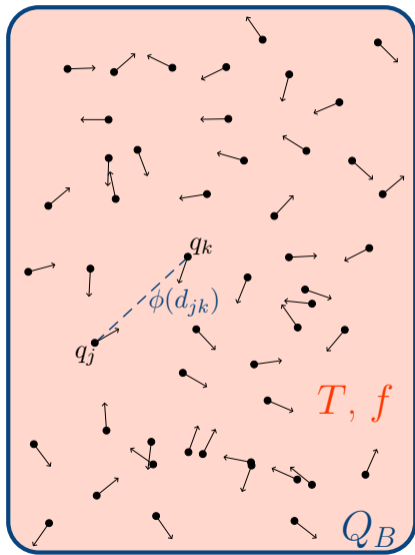
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 - ★ Small β : noise-dominated, weaker correlations.
- In equilibrium, the stationary distribution reads

$$P_{\text{eq}}(\mathbf{r}_1, \dots, \mathbf{r}_N; \beta) \propto e^{-\beta W_N(\mathbf{r}_1, \dots, \mathbf{r}_N)},$$

$W_N(\mathbf{r}_1, \dots, \mathbf{r}_N)$: total Energy of the system.

Stochastic Langevin equation:

$$\mathbf{0} = -f\mathbf{v}_k - \nabla_k W_N + \boldsymbol{\eta}_k(t) \quad (1)$$

White noise:

$$\langle \boldsymbol{\eta}_k(t) \rangle = 0, \quad \langle \boldsymbol{\eta}_k(t) \boldsymbol{\eta}_{k'}(t') \rangle = 2k_B T f \delta_{kk'} \delta(t - t').$$

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Fokker-Planck equation:

$$\partial_t P_\beta^{(N)} = \sum_{k=1}^N \partial_k \left(\partial_k W_N + \frac{1}{\beta} \partial_k \right) P_\beta^{(N)} \quad (2)$$

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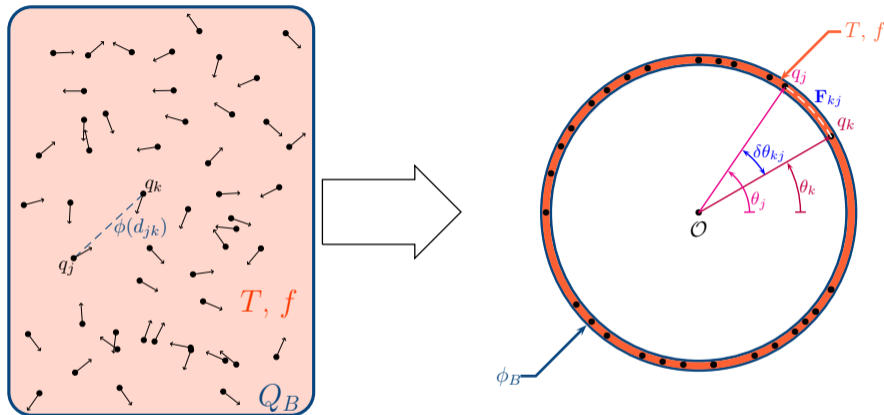
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With $P_\beta^{(N)}(\mathbf{r}_1, \dots, \mathbf{r}_N; t) = e^{-\frac{1}{2}\beta W_N} \Psi_\beta^{(N)}(\mathbf{r}_1, \dots, \mathbf{r}_N; t)$, the **Autoadjoint Schrödinger equation**:

$$\partial_t \Psi_\beta^{(N)} = \sum_{k=1}^N \left(-\frac{\beta}{4} (\partial_k W_N)^2 + \frac{1}{2} \partial_k^2 W_N \right) \Psi_\beta^{(N)} + \frac{1}{\beta} \sum_{k=1}^N \partial_k^2 \Psi_\beta^{(N)} \quad (3)$$



$$\frac{d\theta_k}{d\tau} = \frac{1}{2} \sum_{k' \neq k=1}^N \hat{q}_k \hat{q}_{k'} \cot \left(\frac{\theta_k - \theta_{k'}}{2} \right) + \sqrt{\frac{2}{\beta}} \zeta_k(\tau)$$

$$\begin{aligned} \langle \zeta_k(\tau) \rangle &= 0 \\ \langle \zeta_k(\tau) \zeta_{k'}(\tau') \rangle &= \delta_{kk'} \delta(\tau - \tau') \end{aligned}$$

Associated Schrödinger equation:

$$\partial_\tau \Psi_\beta^{(N)}(\boldsymbol{\theta}, \tau) = \left[E_0(N, \beta) + H_{\text{Int}}(\boldsymbol{\theta}, N, \beta) + \frac{1}{\beta} \sum_{k=1}^N \partial_k^2 \right] \Psi_\beta^{(N)}(\boldsymbol{\theta}, \tau),$$

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Thus, our target equation with $\beta = 2$ will be:

$$\partial_\tau \Psi_{\beta=2}^{(N)}(\boldsymbol{\theta}, \tau;) := \partial_\tau \Psi^{(N)}(\boldsymbol{\theta}, \tau) = \left[\frac{1}{24} N(N^2 - 1) + \frac{1}{2} \sum_{k=1}^N \partial_k^2 \right] \Psi^{(N)}(\boldsymbol{\theta}, \tau) \quad (4)$$

Using the ansatz $\Psi^{(N)}(\boldsymbol{\theta}, \tau) = e^{-E\tau} \Phi^{(N)}(\boldsymbol{\theta})$:

$$E_{n_1, \dots, n_N} = \frac{1}{2} \sum_{k=1}^N n_k^2 - \frac{1}{24} N(N^2 - 1) \rightarrow \Psi^{(N)}(\boldsymbol{\theta}, \tau) = \frac{1}{\sqrt{N!}} \sum_{n_1, \dots, n_N} C_{n_1, \dots, n_N} e^{-E_{n_1, \dots, n_N} \tau} |\Phi^{(N)}(\boldsymbol{\theta})| \quad (5)$$

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Finally, for the Fokker-Planck equation:

$$P^{(N)}(\boldsymbol{\theta}, \tau) = \frac{1}{\sqrt{N!}} Q_N(\boldsymbol{\theta}) \sum_{n_1, \dots, n_N} C_{n_1, \dots, n_N} e^{-E_{n_1, \dots, n_N} \tau} |\Phi^{(N)}(\boldsymbol{\theta})|, \quad (6)$$

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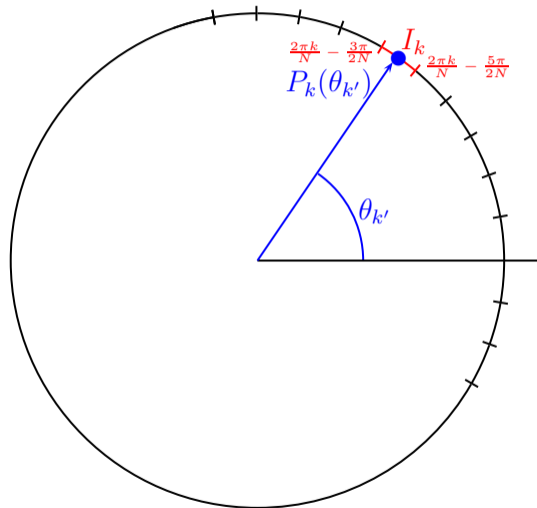
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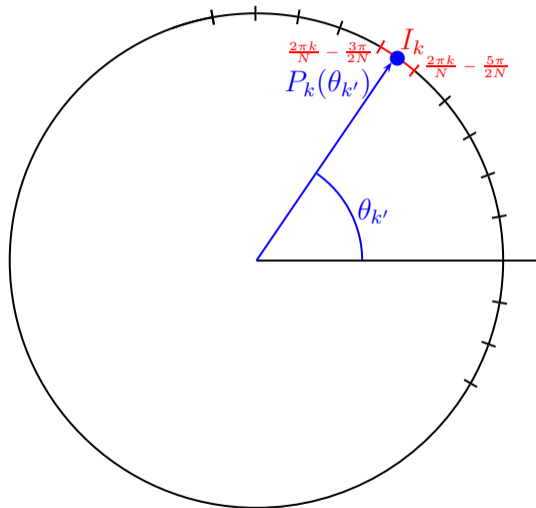
The Boltzmann factor:

$$e^{-W_N} := Q_N(\boldsymbol{\theta}) = 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N-1} \prod_{k'=k+1}^N \sin\left(\frac{\theta_k - \theta_{k'}}{2}\right), \quad (7)$$



- Individual intervals:

$$I_k = \left[\frac{2\pi k}{N} - \frac{5\pi}{2N}, \frac{2\pi k}{N} - \frac{3\pi}{2N} \right], \quad 1 \leq k \leq N.$$

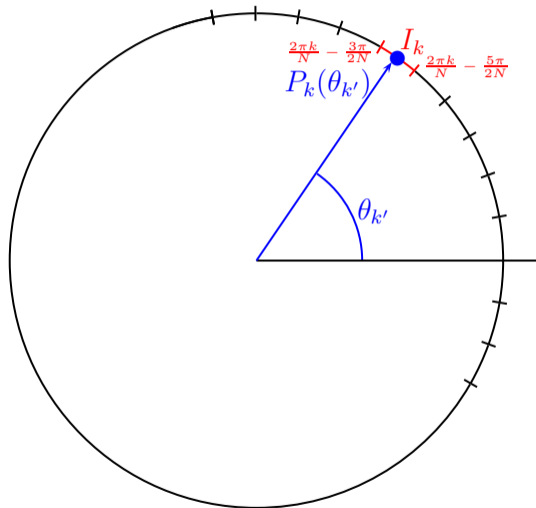


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- Individual probability to place a single particle in an interval:

$$P_k(\theta_{k'}) = \begin{cases} \frac{N}{\pi}, & \text{if } \theta_{k'} \in I_k, \\ 0, & \text{i.o.c.} \end{cases} \quad (8)$$



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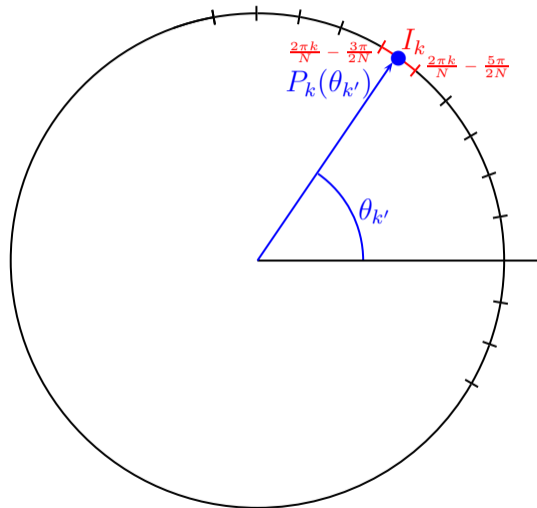
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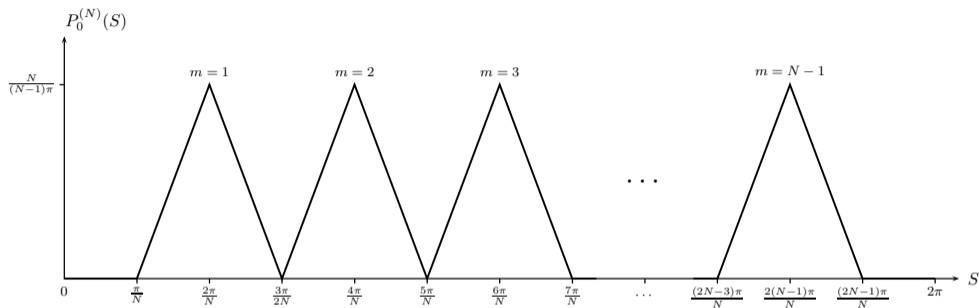
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- Joint probability for all particles:

$$P_0^{(N)}(\boldsymbol{\theta}) = \frac{1}{N!} \sum_{j=1}^{N!} \prod_{k'=1}^N P_{\sigma_j(k')}(\theta_{k'}) \quad (9)$$



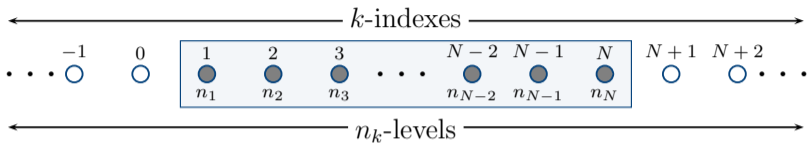


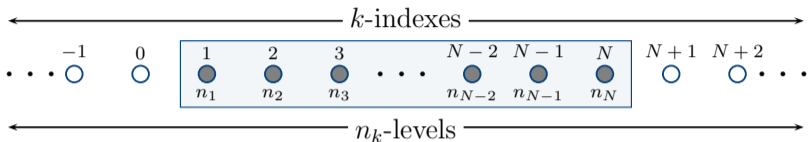
$$\langle S \rangle_0 = \pi, \quad \langle S^2 \rangle_0 = \frac{1 + (4N - 1)^2}{12N^2} \pi^2,$$

$$\sigma_0^2(N) = \frac{2(N - 1)^2 - 1}{6N^2} \pi^2$$

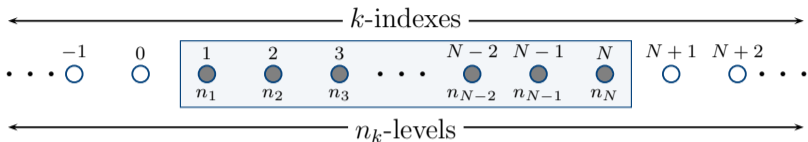
where $\sigma_0^2(N) = \langle S^2 \rangle_0 - \langle S \rangle_0^2$.

Ground state and stationary probability density

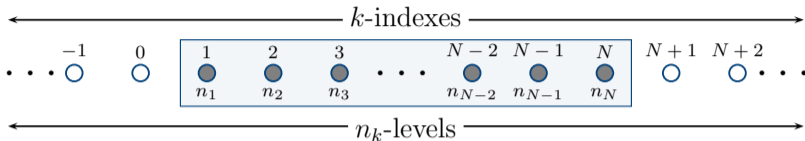




- $n_k = n_1 + k - 1, k \in \mathcal{I}_N^{\text{GS}} = \{1, 2, \dots, N\}, n_1 = -\frac{N-1}{2}$.

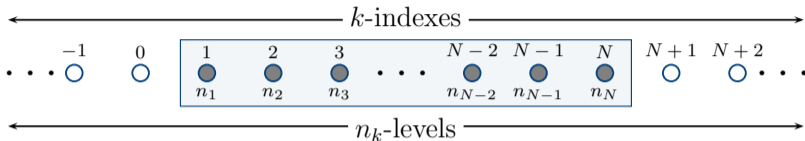


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- Vandermonde determinant for spatial wave functions:

$$|\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})| = \left| \frac{1}{\sqrt{2\pi}} e^{-in_k \theta_{k'}} \right|_{1 \leq k, k' \leq N} = \frac{1}{(\sqrt{2\pi})^N} \prod_{k'=1}^N e^{-in_1 \theta_{k'}} |e^{-i(k-1)\theta_{k'}}|_{1 \leq k, k' \leq N} = \frac{i^{\frac{N(N-1)}{2}}}{(\sqrt{2\pi})^N} Q_N(\boldsymbol{\theta}), \quad (10)$$



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- Probability density function for ground state:

$$P_{\text{GS}}^{(N)}(\boldsymbol{\theta}) = \frac{1}{N!} |\Phi_{\text{GS}}^{\dagger(N)}(\boldsymbol{\theta})| |\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})| = \frac{1}{N!} |\mathcal{K}(\theta_k, \theta_{k'})|. \quad (11)$$

- 2-point correlation distribution:

$$R_{\text{GS}}^{(N)}(\theta_1, \theta_2) = |\mathcal{K}(\theta_1, \theta_2)|^{(2)} .$$

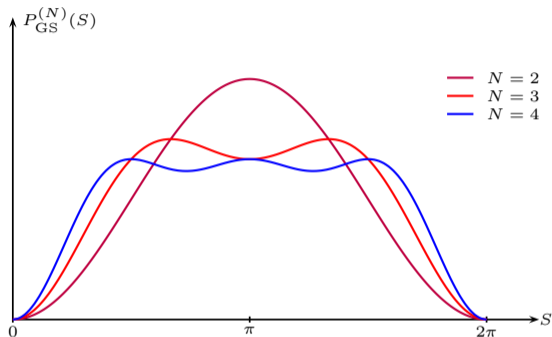
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- Using $S = \theta_2 - \theta_1$:

$$g_{\text{GS}}^{(N)}(S) = \frac{1}{2\pi N(N-1)} \left[N^2 - \frac{\sin^2(NS/2)}{\sin^2(S/2)} \right]$$

Ground state and stationary 2-point correlation distribution



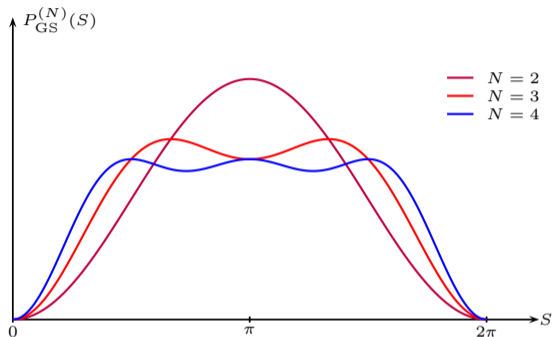
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Ground state and stationary 2-point correlation distribution



- Important fact: *Matrix kernel*

$$K_{kk'} = \sum_{j=1}^N \phi_j^*(\theta_k) \phi_j(\theta_{k'}) = \frac{1}{2\pi} \sum_{j=1}^N e^{inj(\theta_k - \theta_{k'})} = \frac{1}{2\pi} \frac{\sin\left(\frac{N(\theta_k - \theta_{k'})}{2}\right)}{\sin\left(\frac{\theta_k - \theta_{k'}}{2}\right)}, \quad K_{kk} = \frac{N}{2\pi}. \quad (12)$$

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Target:

$$P^{(N)}(\boldsymbol{\theta}, \tau) = \frac{1}{N!} \sum_{n_1, \dots, n_N} \hat{c}_{n_1, \dots, n_N} e^{-E_{n_1, \dots, n_N} \tau} |\Phi_{\text{GS}}^{\dagger(N)}(\boldsymbol{\theta}) \Phi^{(N)}(\boldsymbol{\theta})|, \quad (13)$$

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$$\hat{c}_{n_1, \dots, n_N} := \int_{[0, 2\pi]^N} d^N \boldsymbol{\theta} P_0^{(N)}(\boldsymbol{\theta}) \frac{|\Phi^{\dagger(N)}(\boldsymbol{\theta})|}{|\Phi_{\text{GS}}^{\dagger(N)}(\boldsymbol{\theta})|}. \quad (14)$$

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$$P^{(N)}(\boldsymbol{\theta}, \tau) = \frac{1}{N!} \sum_{n_1, \dots, n_N} \hat{c}_{n_1, \dots, n_N} e^{-E_{n_1, \dots, n_N} \tau} |\Phi_{\text{GS}}^{\dagger(N)}(\boldsymbol{\theta}) \Phi^{(N)}(\boldsymbol{\theta})|, \quad (13)$$

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Problem: Configuration of energy levels, $\{n_k\}_N$, compatible with fixed energy value E .

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$$\text{1st Excited state: } E_{1\text{st}} = \frac{N}{2}.$$

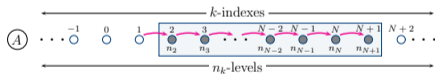
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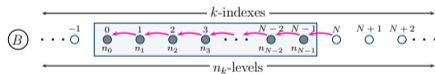
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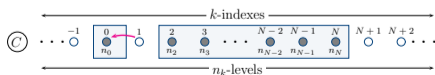
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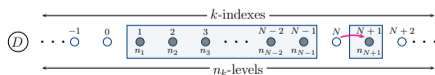
(q) All particles move one step to the right.



(r) All particles move one step to the left.



(s) First particle moves one step to the left.



(t) Last particle moves one step to the right.

- Spatial part:

$$|\Phi_A^{(N)}(\boldsymbol{\theta})| = \prod_{k'=1}^N e^{-i\theta_{k'}} |\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})|, \quad |\Phi_B^{(N)}(\boldsymbol{\theta})| = \prod_{k'=1}^N e^{i\theta_{k'}} |\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})|, \quad (15)$$

$$|\Phi_C^{(N)}(\boldsymbol{\theta})| = (-1)^{\frac{N(N-1)}{2}} |\Phi_D^{\dagger(N)}(\boldsymbol{\theta})| = \sum_{k'=1}^N e^{i\theta_{k'}} |\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})|, \quad |\Phi_D^{(N)}(\boldsymbol{\theta})| = \sum_{k'=1}^N e^{-i\theta_{k'}} |\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})|. \quad (16)$$

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- Associated coefficients:

$$\hat{c}_A = \hat{c}_B = (-1)^{N+1} \left(\frac{2N}{\pi}\right)^N \sin^N\left(\frac{\pi}{2N}\right), \quad \hat{c}_C = \hat{c}_D = 0. \quad (17)$$

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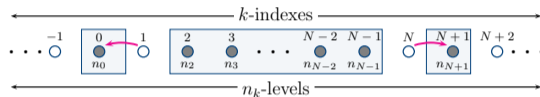
- Finally, contribution of the 1st excited state:

$$P_{\text{1st}}^{(N)}(\boldsymbol{\theta}, \tau) = w_N e^{-\frac{N}{2}\tau} \cos\left(\sum_{k'=1}^N \theta_{k'}\right) P_{\text{GS}}^{(N)}(\boldsymbol{\theta}), \quad w_N := (-2)^{N+1} \left(\frac{N}{\pi}\right)^N \sin^N\left(\frac{\pi}{2N}\right). \quad (18)$$

2nd Excited state: $E_{2\text{nd}} = N - 1$. Has no contribution for $N \neq 2$.

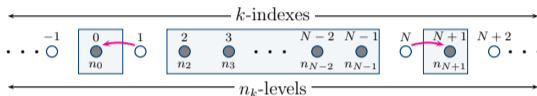
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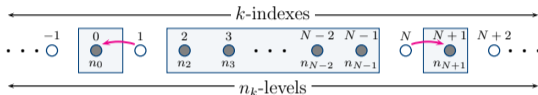


- Spatial part:

$$|\Phi_{3\text{rd}}^{(N)}(\boldsymbol{\theta})| = \left(\sum_{k'=1}^N \sum_{k=1}^N e^{i(\theta_{k'} - \theta_k)} - 1 \right) |\Phi_{\text{GS}}^{(N)}(\boldsymbol{\theta})|, \quad \hat{c}_{3\text{rd}} = N \left[1 - \left(\frac{2N}{\pi} \right)^2 \sin^2 \left(\frac{\pi}{2N} \right) \right] - 1, \quad (19)$$

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- 3rd excited state contribution:

$$P_{3\text{rd}}^{(N)}(\boldsymbol{\theta}, \tau) = f_N e^{-N\tau} \left(\sum_{k'=1}^N \sum_{k=1}^N e^{i(\theta_{k'} - \theta_k)} - 1 \right) P_{\text{GS}}^{(N)}(\boldsymbol{\theta}), \quad f_N := \left[N \left(1 - \left(\frac{2N}{\pi} \right)^2 \sin^2 \left(\frac{\pi}{2N} \right) \right) - 1 \right]. \quad (20)$$

- Probability density function:

$$P^{(N)}(\boldsymbol{\theta}, \tau) = \frac{1}{N!} \left[1 + w_N e^{-\frac{N}{2}\tau} \cos \left(\sum_{k'=1}^N \theta_{k'} \right) + f_N e^{-N\tau} \left(\sum_{k'=1}^N \sum_{k=1}^N e^{i(\theta_{k'} - \theta_k)} - 1 \right) \right] |\mathcal{K}(\theta_k, \theta_{k'})|. \quad (21)$$

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$$R_2^{(N)}(\theta_1, \theta_2) := N(N-1) \int_{[0, 2\pi]^{N-2}} d\theta_3 \cdots d\theta_N P^{(N)}(\boldsymbol{\theta}). \quad (22)$$

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$$g^{(N)}(S, \tau) = \frac{1}{2\pi N(N-1)} \left[N^2 - \frac{\sin^2 \left(\frac{NS}{2} \right)}{\sin^2 \left(\frac{S}{2} \right)} \right] + \frac{2}{\pi N(N-1)} f_N e^{-N\tau} \sin \left(\frac{N+1}{2} S \right) \sin \left(\frac{N-1}{2} S \right) \quad (23)$$

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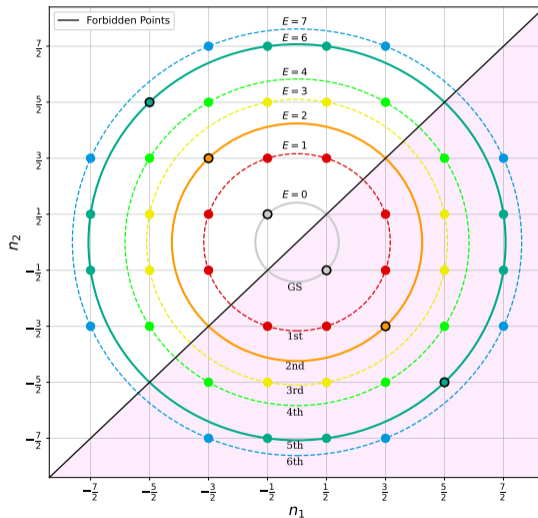
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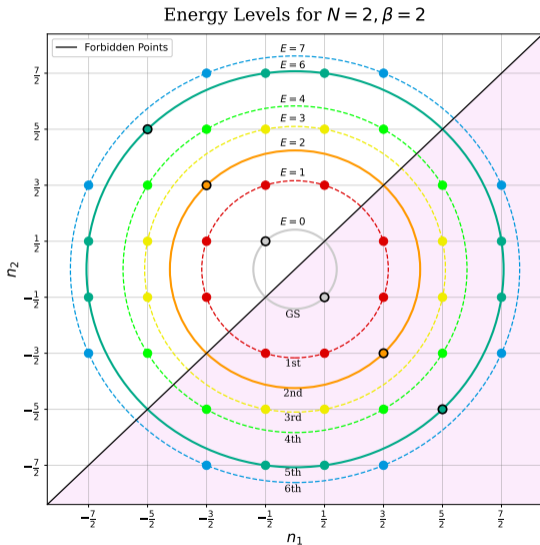
$$\sigma_N^2(\tau) = \frac{\pi^2}{3} - \frac{4}{N(N-1)} \sum_{k=1}^{N-1} \frac{N-k}{k^2} + \frac{4(N+1)}{N^3} f_N e^{-N\tau} \quad (24)$$

Energy Levels for $N = 2, \beta = 2$



- Using same ideas:

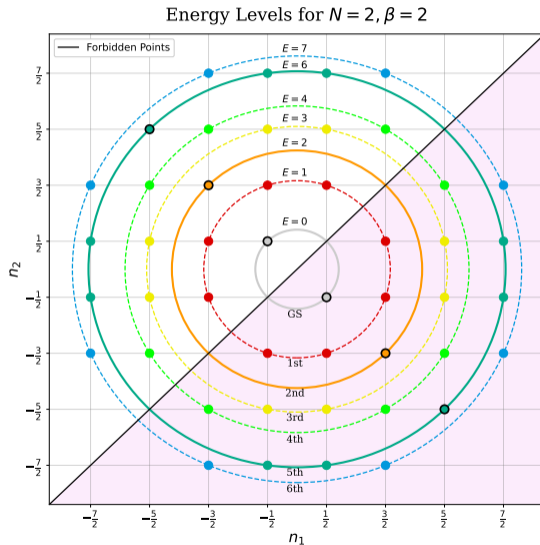
$$g^{(2)}(S, \tau) = -\frac{1}{\pi} \sin\left(\frac{S}{2}\right) \sum_{k=1}^{\infty} \hat{c}_k e^{-k(k-1)\tau} \sin\left(\frac{2k-1}{2}S\right),$$



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$$\hat{c}_k = -\frac{4}{\pi^2} \int_{-\pi/4}^{\pi/4} \int_{3\pi/4}^{5\pi/4} d\theta_1 d\theta_2 \frac{\sin\left(\frac{(2k-1)(\theta_2-\theta_1)}{2}\right)}{\sin\left(\frac{(\theta_2-\theta_1)}{2}\right)},$$

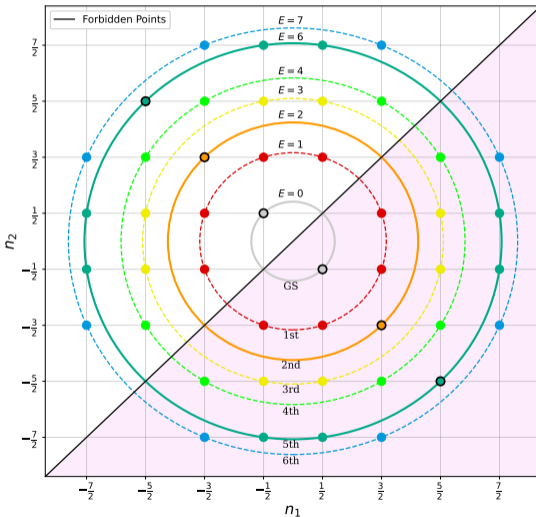


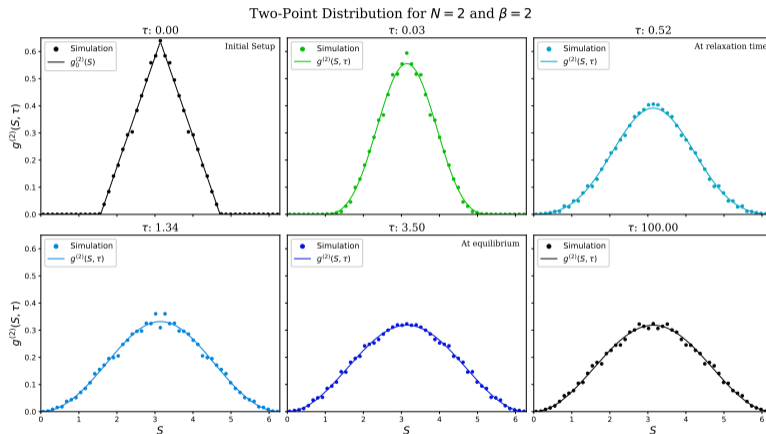
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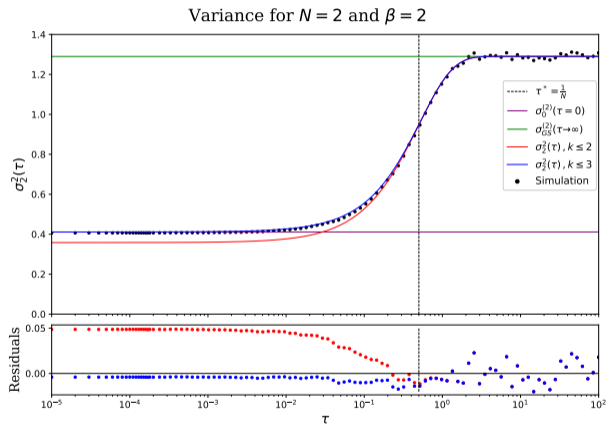
$$\sigma_2^2(\tau) = \frac{\pi^2}{3} - 2 + \frac{16}{\pi^2} \sum_{k=2}^{\infty} \tilde{c}_k e^{-k(k-1)\tau} \frac{2k-1}{k^2(k-1)^2}.$$

Energy Levels for $N = 2, \beta = 2$ 

For $k = 1, 2$ and 3 (Up to 5th excited state).


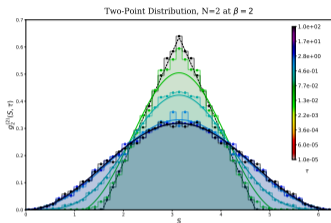
$$g^{(2)}(S, \tau) = \frac{1}{\pi} \sin^2 \left(\frac{S}{2} \right) + \frac{\pi^2 - 16}{\pi^3} e^{-2\tau} \sin \left(\frac{S}{2} \right) \sin \left(\frac{3S}{2} \right) + \frac{\pi^2 - 8}{\pi^3} e^{-6\tau} \sin \left(\frac{S}{2} \right) \sin \left(\frac{5S}{2} \right).$$

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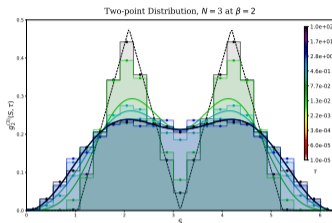


$$\sigma_2^2(\tau) = \frac{\pi^2}{3} - 2 + \frac{3(\pi^2 - 16)}{2\pi^2} e^{-2\tau} + \frac{5(\pi^2 - 8)}{18\pi^2} e^{-6\tau}.$$

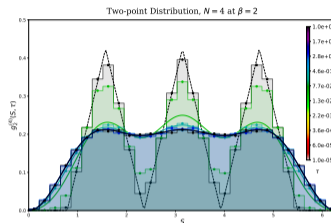
2-point correlation distribution up to 3rd excited state



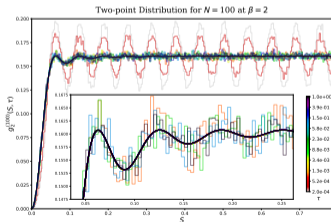
(u) For $N = 2$.



(w) For $N = 3$.

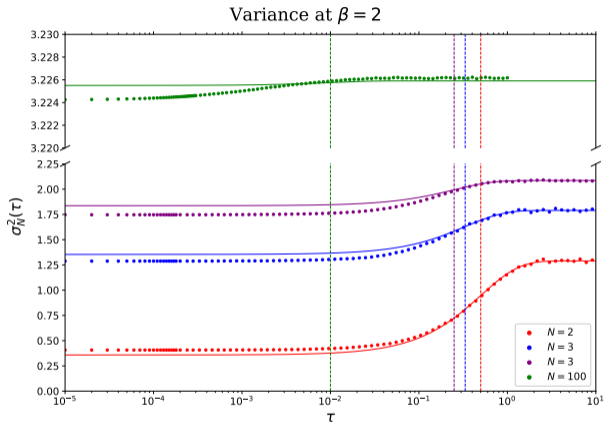


(x) For $N = 4$.



(v) For $N = 100$.

$$g^{(N)}(S, \tau) = \frac{1}{2\pi N(N-1)} \left[N^2 - \frac{\sin^2\left(\frac{NS}{2}\right)}{\sin^2\left(\frac{S}{2}\right)} \right] + \frac{2}{\pi N(N-1)} f_N e^{-N\tau} \sin\left(\frac{N+1}{2}S\right) \sin\left(\frac{N-1}{2}S\right).$$



$$\sigma_N^2(\tau) = \frac{\pi^2}{3} - \frac{4}{N(N-1)} \sum_{k=1}^{N-1} \frac{N-k}{k^2} + \frac{4(N+1)}{N^3} \left[N-1 - \frac{4N^3}{\pi^2} \sin^2 \left(\frac{\pi}{2N} \right) \right] e^{-N\tau}.$$

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Next steps

- Explore $\beta \neq 2$ regimes for relaxation dynamics via perturbation theory.
- Different initial conditions and quenches.

May the Force be with you!!



Prepare simulation process:

General Parameters: N_0 , initial number of particles; δt , time step; N_{Iter} , total iterations to perform and N_{Sim} , total number of simulations.

Control Parameters: $\beta = \frac{q^2}{k_B T}$, charge-temperature ratio; l_f , scaled fusion length and p_f probability of make a fusion.

Dynamic Quantities of Interest: $N(t)$, particle number evolution; $c_k(t)$, charge-clusters evolution and $g^{(1)}(\theta, t)$, first neighbors correlation distances.

Simulation scheme:

1. Random location of the initial particles on the unit circle.
2. Evolve particles using *discrete* Over-damped Langevin Eq. of Motion:

$$\delta\theta_k = \frac{q_k}{4\pi\epsilon_0} \frac{\delta t}{f} \sum_{j \neq k} q_j \cot\left(\frac{\theta_j - \theta_k}{2}\right) + \frac{1}{f} \mathcal{N}_k(t)$$

- Get distances to first neighbors and test:
 - If $\Delta\theta_{ij} < \theta_f$ AND $p \leq p_f$, THEN make a Fusion.
 - Store data: $N(t), q_k(t), \Delta\theta_{ij}$
- Repeat from step 2 until:
 - $N = 1$.
 - $t = N_{\text{Iter}}\delta t$.

Numerical Values:

- $N_0 = 100$ located at unitary circle.
- $N_{\text{Sim}} = 1000$ per set $\{N, N_{\text{Iter}}, \beta, l_f, p_f\}$.
- $\theta_f = \frac{l_f}{10} \frac{2\pi}{N_0}$.
- $\beta \in [0.1, 8.0]$
- $l_f \in [0.1, 10.0]$
- $p_f \in [0.1, 1.0]$

- For δt : Variance for small jumps:

$$\langle (\delta\theta_k)^2 \rangle = \frac{2\delta t}{\beta f}.$$

- Constraint for diffusive process:

$$\sigma = \sqrt{\langle (\delta\theta_k)^2 \rangle} = \sqrt{\frac{2\delta t}{\beta f}} \ll \theta_f = \frac{l_f}{10} S_{N_0}.$$

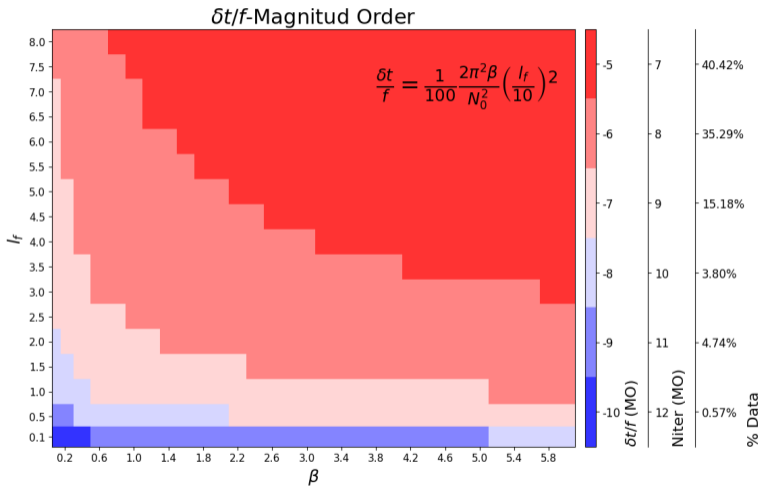
- Thus, for δt :

$$\delta t \ll \frac{2\pi^2\beta f}{N_0^2} \left(\frac{l_f}{10}\right)^2.$$

- Numerically:

$$\frac{\delta t}{f} = \frac{1}{100} \frac{2\pi^2\beta}{N_0^2} \left(\frac{l_f}{10}\right)^2$$

- $N_{\text{Iter}} = \frac{10^2}{\delta t}$.



Useful form for Vandermonde determinant:

$$\det(\mathbb{V}) = \prod_{k=1}^m \prod_{k'=k+1}^N (z_{k'} - z_k) \det \left(\sum_{j=1}^{k-m} z_{k'}^{k-m-j} \cdot e_{j-1}^{(m)}(z) \right)_{\substack{m+1 \leq k' \leq N, \\ k \geq m+1}}^{(N-m)}, \quad (25)$$

where m is the number of deleted rows and columns, and

$$e_n^{(m)}(z) := e_n(z_1, z_2, \dots, z_m) = \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_m} z_{j_1} \cdot z_{j_2} \cdot \dots \cdot z_{j_m}, \quad e_0^{(m)}(z) := 1. \quad (26)$$

Alternative form for determinant function:

$$|\mathcal{K}^{(N)}(\theta_k, \theta_{k'})|^{(m)} = \sum_{j=1}^{(m-1)!} \epsilon(\sigma_j) \prod_{k=1}^{m-1} K_{k, \sigma_j(k)} K_{m, m} - \sum_{k'=1}^{m-1} \sum_{j=1}^{(m-1)!} \epsilon(\sigma_j) \prod_{k=1, k \neq k'}^{m-1} K_{k, \sigma_j(k)} K_{k', m} K_{m, \sigma_j(k')}. \quad (27)$$

Furthermore:

$$\int_0^{2\pi} d\theta_3 e^{-i\theta_3} K_{k3} K_{3k'} = e^{-i\theta_{k'}} \hat{K}_{kk'}, \quad \int_0^{2\pi} d\theta_3 \hat{K}_{k3} K_{3k'} = \hat{K}_{kk'}, \quad \int_0^{2\pi} d\theta_3 e^{i\theta_3} K_{k3} \hat{K}_{3k'} = e^{i\theta_k} \hat{K}_{kk'}, \quad (28)$$

$$\hat{K}_{kk'} = \frac{1}{2\pi} \sum_{j=1}^{N-1} e^{in_j(\theta_k - \theta_{k'})}. \quad (29)$$