

NON EQUILIBRIUM MACROSCOPIC COMPLEX SYST

MANY ADJECTIVES

NON-EQ: WE'LL TALK ABOUT OUT OF EQ
DYNAMICS

SOMETIMES

QUANTUM: WELL... I'LL GO FROM CLASSICAL TO
QUANTUM BACK & FORTH
NOT SO MANY DIFFERENCES, THOUGH
I'LL HIGHLIGHT THEM

MACROSC: VS. SINGLE PARTICLE. MANY-BODY
THERMODYN LIMIT IMPORTANT $\Rightarrow$
LAW OF LARGE NUMBERS
STAT PHYS.

COMPLEX: HARD, NOT EASY $\Rightarrow$ INTERESTING

Plan

1 - INTRODUCTION TO STOCHASTIC PROCESSES

- LANGENIN EQS.

JUSTIFICATION

ORIGINAL - GENERAL

EXTENSIONS

APPLICATIONS

1st LECT.

- FOKKER-PLANCK.
- GEN. FUNCT.

2nd LECT.

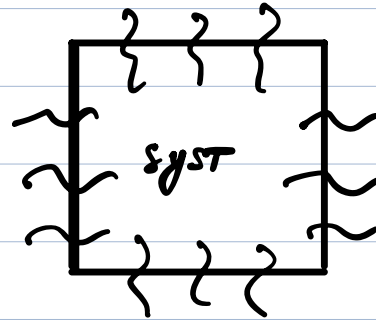
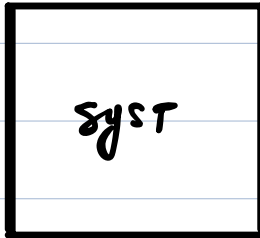
2 - SOME FEATURES OF MACROSC. SYST. OUT OF EQ.

3rd LECT.

CLOSED

VS.

OPEN



CLASS. NOTATION

SURROUNDINGS

ENVIRONMENT - BATH

$$E = H(\{\vec{x}_i, \vec{p}_i\}) = \text{CST.}$$

E FLUCTUATES

MAY $\exists$ OTHER CONSERVED
QUANTITIES, EG MOMENTUM,
ANG MOMENTUM, ETC.

EXCHANGES $\neq$
BATH ALSO N
OR SOMETHING ELSE

MICROCANONICAL

CANONICAL

I WILL NOT TALK
ABOUT THESE BUT
ALSO VERY INTERESTING

GRAN. CANONICAL

EQUILIBRIUM MEASURES

MICROCANONICAL - CLOSED

$$P(\{\vec{x}, \vec{p}\}) = \frac{1}{N_c} \delta(\mathcal{H}(\{\vec{x}, \vec{p}\}) - E)$$

$N_c \neq \text{CT. MOTION}$ $\mathcal{H}_\mu(\{\vec{x}, \vec{p}\})$ $\mu = 1, \dots, N_c$ $\frac{d\mathcal{H}_\mu}{dt} = 0$

$\mathcal{H}_\mu(0)$ THEIR INITIAL VALUES

CANONICAL - OPEN

$$P(\{\vec{x}, \vec{p}\}) = \frac{1}{Z(\{\gamma_\mu\})} e^{-\sum_{\mu=1}^{N_c} \gamma_\mu \mathcal{H}_\mu(\{\vec{x}, \vec{p}\})}$$

eg. $\mathcal{H}_1 = H$, $\gamma_1 = \beta$ INVERSE TEMP
 $\mathcal{H}_2 = N$, $\gamma_2 = \mu$ CHEMICAL POT.

DYNAMICS

CLOSED

NEWTON - HAMILTON

CLASS

HEISENBERG - SCHRÖDINGER

QUANTUM

OPEN

LANGVIN SINGLE NOISY TRAJECTORIES

FOKKER-PLANCK - MASTER EQ $P(\{\vec{x}_i, \vec{p}_i\}, t)$

MSR-JD GENERATING FUNCT TRAJ.

LINDBLAD EQ.

MASTER EQ. FOR $\hat{\rho}$

SCHWINGER-KELDysh CLOSED TIME PATH WITH

INFLUENCE FUNCTIONALS IN GEN. FUNCT

GEN MSR-JD

LANGERIN'S LANGERIN EQ.

COLLOID IN



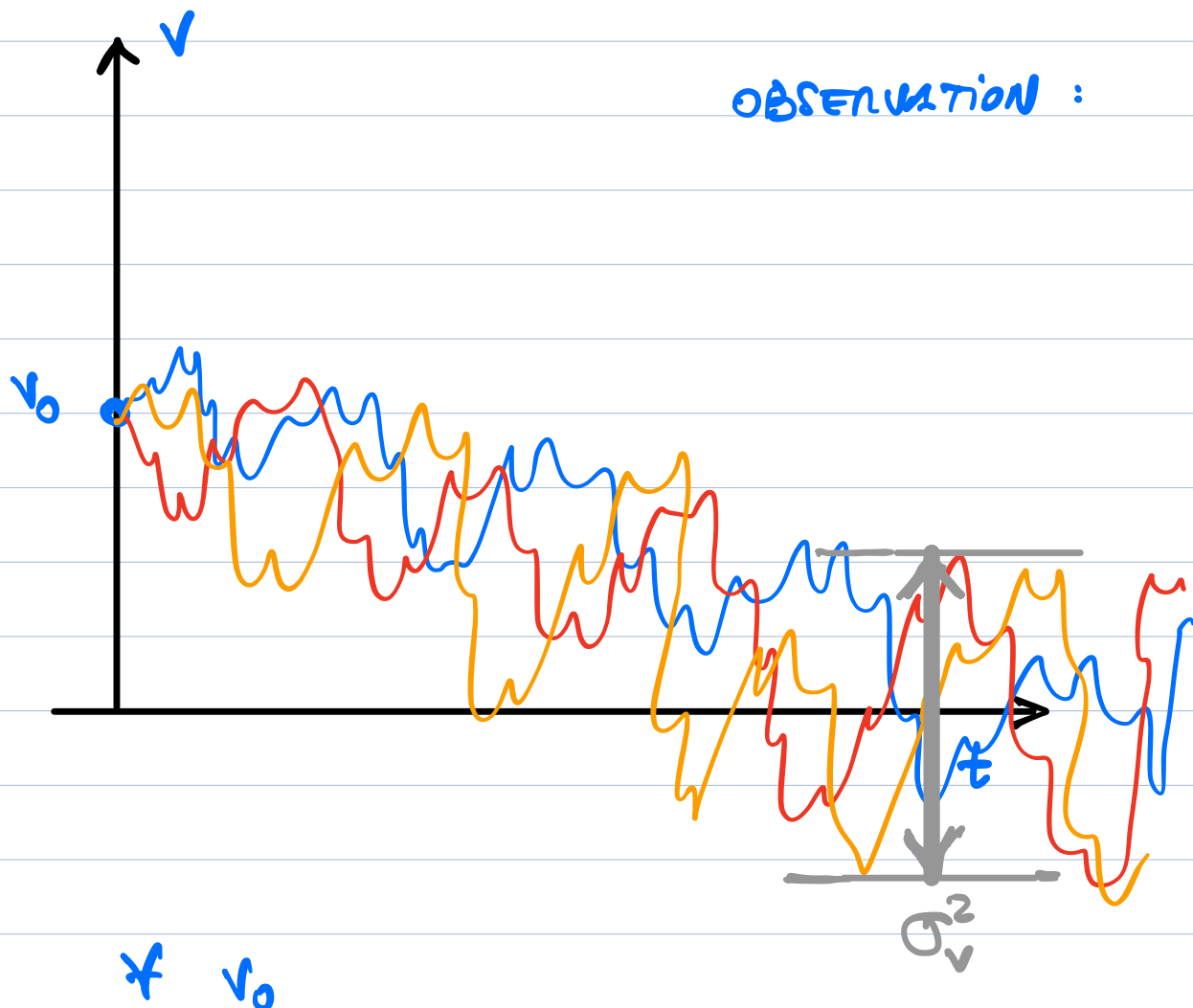
BROWN'S EXP

POWEN

ALIVE OR DEATH

- MANY MANY COLLISIONS BTW WATER MOLECULES & COLLOID IN UNIT TIME (MEASUREMENT CLOCK, eg. 1sec)

- EFFECTS : FRICTION & NOISE



ALL "TRAJECTORIES" ARE $\neq$
BUT THEY SHARE STATIST.
PROPERTIES

SO THAT

NEWTON'S LAW

$$m \ddot{x} = F_{\text{ENV}} \quad d=1$$

$$= -\gamma \dot{x} + \xi$$

↑ FRICTION COEFF

← RANDOM FORCE

WILL SLOW DOWN THE
PART

RANDOM FORCE

$\xi(t)$ GAUSSIAN, JUST NEED TO KNOW
AVEN & CORR. IN TIME

CHOSEN SO THAT :

$$\langle v(t) \rangle \longrightarrow 0$$

FORGET i.c.

$$\frac{m}{2} \langle v^2(t) \rangle \longrightarrow \frac{k_B T}{2}$$

EQUIPART.
AT AMBIENT
TEMP

$$\langle S(t) \rangle = 0$$

$$t > t_2 = m/\gamma$$

$$\langle S(t) S(t') \rangle = 2\gamma k_B T \delta(t-t')$$

AVER $\langle \dots \rangle$ MEANS OVER REPETITION
OF EXP.

IMPORTANT:

- SEPARATION OF TIME SCALES
BATH MUCH FASTER THAN SYST
- SEPARATION OF SIZES
BATH MUCH LARGER THAN SYST

WE USE LANGEVIN EQS IN VERY $\neq$ CASES
WHY? 30' 8

JUSTIFICATION OF NOISE / DISSIP. IN LANGEVIN EQ.

CLASSICAL LANGEVIN. BROWNIAN MOTION, THEN GENER.

LANGEVIN ~ 1900

BETTER & MORE GENERAL:

MODEL OF SYST H_{SYST} BUT ALSO
MODEL OF BATH & COUPLING

PHONONS, SIMPLEST IDEA

HARMONIC OSCILLATORS & LINEAR COUPLING

$$H_B = \sum_{a=1}^{N_0} \left(\frac{\pi_a^2}{2m_a} + \frac{m_a \omega_a^2}{2} q_a^2 \right)$$

$d=1$

TO SIMPLIFY

THE NOTATION

$$H_I = \sum_{a=1}^{N_0} \frac{c_a q_a}{\sqrt{N_0}} \cdot f(x)$$

(COUNTERTERM LATER)

c_a COUPLING CONST. $\mathcal{O}(1)$

$\{q_a, \pi_a\}$ OSCILLATOR'S PHASE SPACE VARIABLES

$N_0 \neq$ OSCILLATORS (WILL GO TO ∞)

SOLVABLE NEWTON/HAM. DYN FOR THE OJC.

LINEAR EQS. FORCED BY THE COUPLING TO x

VIA $f(x)$ USUALLY $f(x) = x$ SIMPLEST CASE
GEN. LATER

SOME VERY SIMPLE CALCULATIONS & USING
THE IMPORTANT ASSUMPTION

$$P(\{q_a(0), \pi_a(0)\})$$

EQUILIBRIUM BATH
CANONICAL BOLZMANN
(COUPLING TO $x(0)$)

TRICKY ISSUE ABOUT A COUNTERTERM TO BE DISCUSSED

TAKING $N_0 \rightarrow \infty$ A BATH: ENORMOUS

GAUSSIAN RANDOM INITIAL CONDITIONS $\Rightarrow$

INDUCE RANDOMNESS / STOCHASTICITY $k_B T$
FRICTION γ

THE DYN NO LONGER CONSERVE ENERGY OR
OTHER CONST MOTION OF THE SYST

$\Rightarrow$

GENERALIZED LAMBERT EQS

$$m\ddot{x}(t) + \int_0^t dt' \Gamma(t-t') \dot{x}(t') = F[x(t)] + S(t)$$

NOISE

$$S(t) = - \sum_{a=1}^{N_0} C_a \left[\frac{\pi_a(0)}{m_a \omega_a} \sin \omega_a t + \left(q_a(0) + \frac{C_a x(0)}{m_a \omega_a^2} \right) \cos \omega_a t \right]$$

$$f(x) = x$$

STATISTICS OF $\{q_a(0), \pi_a(0)\} \Rightarrow$ THE ONE OF ξ
USUAL EQ CHOICE FOR OSCILLATORS: $\leftarrow T(t)$

$$\langle S(t) \rangle = 0$$

$$\langle S(t) S(t') \rangle = k_B T \Gamma(t-t')$$

$\langle \dots \rangle$ MEANS AVERAGE WITH RESPECT TO
THE i.c. OF THE OSC.

WITH

$$\Gamma(t-t') = \frac{1}{N_0} \sum_{a=1}^{N_0} \frac{C_a^2}{m_a \omega_a^2} \cos[\omega_a(t-t')]$$

SYMMETRIC FRICTION KERNEL $t \leftrightarrow t'$

FUNCT FORM DEPENDS ON $\{C_a, m_a, \omega_a^2\}$

PROPS OF OSC. BATH

KAWASAKI, ZWANZIG 60s-70s

usually
$$S(\omega) = \frac{1}{N_0} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a} \delta(\omega - \omega_a)$$

$$\Gamma(t-t') = \int_0^\infty d\omega \frac{S(\omega)}{\omega} \cos \omega(t-t')$$

THE CHARACTERISTICS OF THE OSCILLATORS
 $\Rightarrow$ NOISE, FRICTION PROPS. VIA $\Gamma(t-t')$

FROM CLASSICAL MECHANICS TO STOCHASTIC PROCESSES

REDUCED SYSTEM

NON-MARKOVIAN-MEMORY

SPECTRAL FUNCTION LEADING TO POWER-LAWS

$$\frac{S(\omega)}{\omega} = \frac{2\gamma_0}{\pi} \left(\frac{|\omega|}{\tilde{\omega}} \right)^{\alpha-1} f_c \left(\frac{|\omega|}{\Lambda} \right)$$

USUAL CASES

$$\Gamma(t-t') = \begin{cases} 2\gamma_0 \delta(t-t') & \alpha=1 \\ & \text{WHITE NOISE} \\ \gamma_0 \exp\left(-\frac{|t-t'|}{\tau}\right) & \text{EXP. DECAY} \\ |t-t'|^{-\alpha} & \text{POWER LAW} \\ & \text{SUB/SUPER-OHMIC} \end{cases}$$

ALL THE INFO ABOUT BATH is in $k_B T$, $\Gamma(t-t')$

OVERDAMPED LIMIT - USUAL IN STAT PHYS.

VALID FOR LARGE FRICTION/MASS $\gamma/m \gg 1$

OFTEN TIME SCALE OF MOMENTUM

RELAX. IS VERY SHORT, GAUSSIAN WITH

$$\langle v(t) \rangle \rightarrow v_0 \quad \langle v^2(t) \rangle \rightarrow \frac{k_B T}{m}$$

DRD $m \ddot{x}$ FROM EQ. MOTION

SEPARATION OF TIME-SCALES p AND x
VERY DIFFERENT

$$f(x) \quad \text{GENERIC} \quad H_I = \sum_{a=1}^{N_0} \frac{c_a q_a}{\sqrt{N_0}} f(x)$$

$$m\ddot{x}(t) + f'(x(t)) \int_0^t dt' \Gamma(t-t') f'(x(t')) \dot{x}(t') \\ = F[x(t)] + S(t) f'(x(t))$$

MULTIPLICATIVE NOISE

$$+ K \cdot \underbrace{x}_{f(x)}$$

MENTIONED COUNTER TERM.

TO GET THE EQ. I WROTE, ADD

$$H_C = \frac{1}{2} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} x^2$$

AND CHOOSE
INITIAL DIST OF
 $\{T_a(t), q_a(t)\}$
WITH NO WITHIN

NEEDED TO AVOID A "REMAN." OF H_{Syst}

PHYSICAL CHOICE TO DO IT OR NOT

CONNECTION WITH USUAL

MULT. NOISE WHITE NOISE Lange Eq

$$\dot{x} = \bar{f}(x) + g(x) \xi \quad \text{FOUND IN LITERATURE}$$

TAKE $\Gamma(t-t') = 2\gamma_0 \delta(t-t')$ IN (1)

DROP $m\ddot{x}$: OVERDAMPED LIMIT

SET $\gamma_0 = 1$ BY A REDEF. OF TIME

$$\left(f'(x(t))\right)^2 \dot{x}(t)$$

$$= F[x(t), \lambda(t)] + S(t) f'(x(t))$$

DIVIDE BY $\left(f'(x(t))\right)^2$ AND REDEFINE THE
DETERM FORCE TERM $F/(f')^2 = \bar{f}$
AND

$$g = 1/f'$$

AND THAT'S IT.

MULTI-VARIABLE LANGEVIN EQ.

$$m \ddot{x}(t) + f'(x(t)) \int_0^t dt' \Gamma(t-t') \dot{x}(t') = F[x(t), \lambda(t)] + S(t) f'(x(t)) \quad (1)$$

POSSIBLE TIME-DEP PARAM.

ADD INDICES TO MAKE IT VALID FOR

$$i = 1, \dots, N$$

$$a = 1, \dots, d$$

LOOK AT LTC DEV. PHYS. FOR FULL GENERALITY

INDEPENDENT

USUAL ADDITIVE NOISE CASE, ONE IDENTICAL

BATH PER

$$m \ddot{x}_i^a + \int_0^t dt' \Gamma(t-t') \dot{x}_i^a = \text{d.o.f.}$$

$$= F_i^a [\vec{x}_i(t), \vec{\lambda}_i(t)] + S_i^a(t)$$

$$\langle S_i^a(t) S_j^b(t') \rangle = k_B T \Gamma(t-t') \delta_{ij} \delta^{ab}$$

REFS.

DERIVATION & DISCUSSION OF LANG EQ

<http://www.lpthe.jussieu.fr/~leticia/SEMINARS/Langern.pdf>

IN ADV. PHYS.

SKETCHY SLIDES

<http://www.lpthe.jussieu.fr/~leticia/SEMINARS/ictp-non-Markovian-bis.pdf>

APPLICATIONS

INVERSE PROBLEM

OBSERVE THE MOTION OF A TRACER

eg. BROWNIAN MOTION

CONCLUDE ABOUT THE PROPS OF THE BATH
THE FORCES IT INDUCES ON TRACER

LANGBEIN DID IT THIS WAY, EARLY 1900S

EXPERIMENTS, THEORY



TRACER IN A CELLULAR ENVIRONMENT



HOW TO GO FROM

INFER FROM TRAJECTORIES

P. RONCEY (MARSIEUX)

MIT GROUP EXPS.

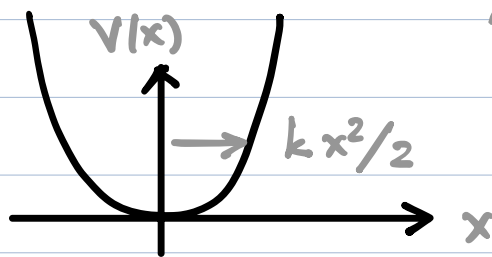
MANY OTHERS

INFERENCE OF BATH - ENVIRONMENT

YANG ET AL 03; MIN, LUO, CHERAYIL, KOU & XIE 05

$$C_x(t, t') = \langle x(t) x(t') \rangle \quad \text{MEASURED}$$

$$= \frac{1}{k} E_{\alpha, 1} \left(- \frac{k |t - t'|^\gamma}{\eta} \right)$$

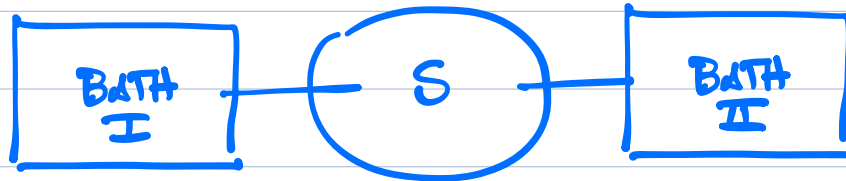


MITTAG LEFFLER - FCT.

$$\gamma \approx 0.5$$

BATH?

SYSTEMS SUBMITTED TO TWO BATHS



LANGERIN EQ FORMULATION

FRICTION $\Gamma_1(t-t') + \Gamma_2(t-t')$

NOISE - NOISE CORR $k_B T_1 \Gamma_1(t-t') +$
 $k_B T_2 \Gamma_2(t-t')$

FOR THE SETTING IN WHICH EACH SYST
VARIABLE COUPLED TO BOTH BATHS.

SYSTEMS FORCED EXTERNALLY BY
A TIME-DEPENDENT DRIVE

$$F[x(t), \lambda(t)]$$

SETTING USED IN THE CONTEXT OF
FLUCT. THEOREMS

ENTROPY PRODUCTION

STOCHASTIC THERMODYNAMICS

ACTIVE EINSTEIN-UHLENBECK PARTICLES

of F. VAN WIJLAND'S
TALKS

OVERDAMPED LIMIT

also my seminar

$$m_0 \dot{x} = F + \eta$$

$$\dot{\eta} = -\frac{1}{\tau} \eta + \xi$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t) \xi(t') \rangle = k_B T \delta(t-t')$$

INTEGRATING η :

$$\eta(t) = \eta(0) e^{-t/\tau} + \int_0^t dt' e^{-(t-t')/\tau} \xi(t')$$

$$\langle \eta(t) \rangle = 0$$

$$\langle \eta(t) \eta(t') \rangle \rightarrow k_B T \tau e^{-(t-t')/\tau}$$

$$\langle \eta(t) \eta(t') \rangle = \int_0^t dt'' \int_0^{t'} dt''' e^{-\frac{(t-t'')}{\tau} - \frac{(t'-t''')}{\tau}} \underbrace{\langle \delta H'' \delta H''' \rangle}_{2 k_B T \delta(t''-t''')}$$

$$t > t' \quad 2 k_B T \delta(t''-t''')$$

$$= 2 k_B T \int_0^{t'} dt'' e^{-(t-t''+t'-t'')/\tau}$$

$$= 2 k_B T e^{-(t+t')/\tau} \int_0^{t'} dt'' e^{2t''/\tau}$$

$$= 2 k_B T \frac{\tau}{2} \left[e^{-(t-t')/\tau} - e^{-(t+t')/\tau} \right]$$

$$\xrightarrow{t' \gg \tau} k_B T \tau e^{-(t-t')/\tau}$$

EXPONENTIAL CORR.

IN TIME OF NOISE τ

PERSISTENCE TIME τ

$$\gamma_0 \frac{[x]}{[t]} = [F] = \frac{[E]}{[x]} \Rightarrow \gamma_0 [x^2] = [k_B T] [t]$$

FROM EQ. ON x (1)

$$\frac{[\eta]}{[t]} = \frac{[\eta]}{[\tau]} = [\xi] \Rightarrow [\eta] = [\xi] [t]$$

FROM EQ. ON η (2)

$$[t] = [\tau] \quad \checkmark$$

$$[\xi^2] = \frac{[k_B T]}{[t]} \quad \text{FROM NOISE \& CORREL. IN (2):}$$

$$[\eta] = \frac{[k_B T]^{1/2}}{[t]^{1/2}} \cdot [t] = [k_B T]^{1/2} [t]^{1/2} \quad (3)$$

$$\text{also } [\eta] = [F] = \frac{[E]}{[x]} = \frac{[k_B T]}{[x]} \quad \text{FROM EQ. ON x}$$

$$\text{USING (1)} \quad [\eta] = \frac{[k_B T]^{1/2} [\gamma_0]^{1/2}}{[t]^{1/2}} \quad (4)$$

$$(3) \text{ \& (4)} \Rightarrow [t] = [\gamma_0]^{1/2}$$

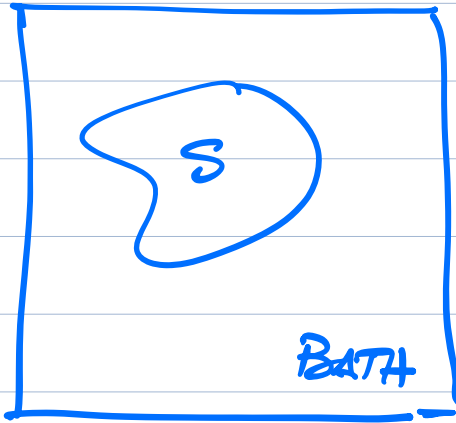
THE NOISE TERM $\int \dot{x}$ AND THE FRICTION TERM $\int_0 \dot{x}$ ARE NOT BALANCED

THE LANG EQ IS NOT OF THE FORM DERIVED ABOVE FOR EQUIL BATHS.

$\Rightarrow$ OUT OF EQUIL. ENVIRONMENT

BACK TO FORMALISM

SUMMARY 1ST LECTURE



INTEGRATE AWAY
BATH
if IN EQUIL.

⇒ GENERALIZED LANG EQ

w/ $m\ddot{x}$, RETARDED FRICTION,
COLOURED NOISE POSSIBLY MULTIP.

SIMPLIFICATIONS

OVERDAMPED $m\ddot{x} \rightarrow \text{DROP}$

MARKOV $\Gamma(t-t') \rightarrow 2\gamma \delta(t-t')$

ADDITIVE NOISE

TODAY : OTHER WAYS OF DEALING
w/ THESE SITUATIONS

IN THEIR MOST GEN. FORM
OR APPROX.

1- FOKKER - PLANCK / KRAMERS
Σ LINDBLAD FOR QUANTUM
MARKOV

2- GEN. FUNCT

MSR-JD

SCHWINGER - DYSON (CTP)

FEYNMAN VERNON

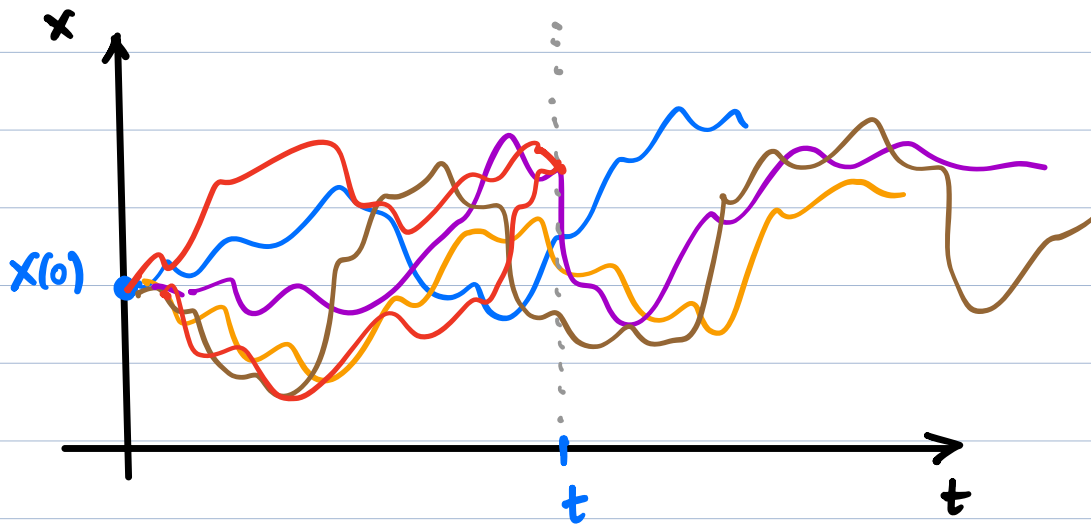
GENERIC

KRAMERS / FOKKER-PLANCK EQUATION

$$P(x, p; t)$$

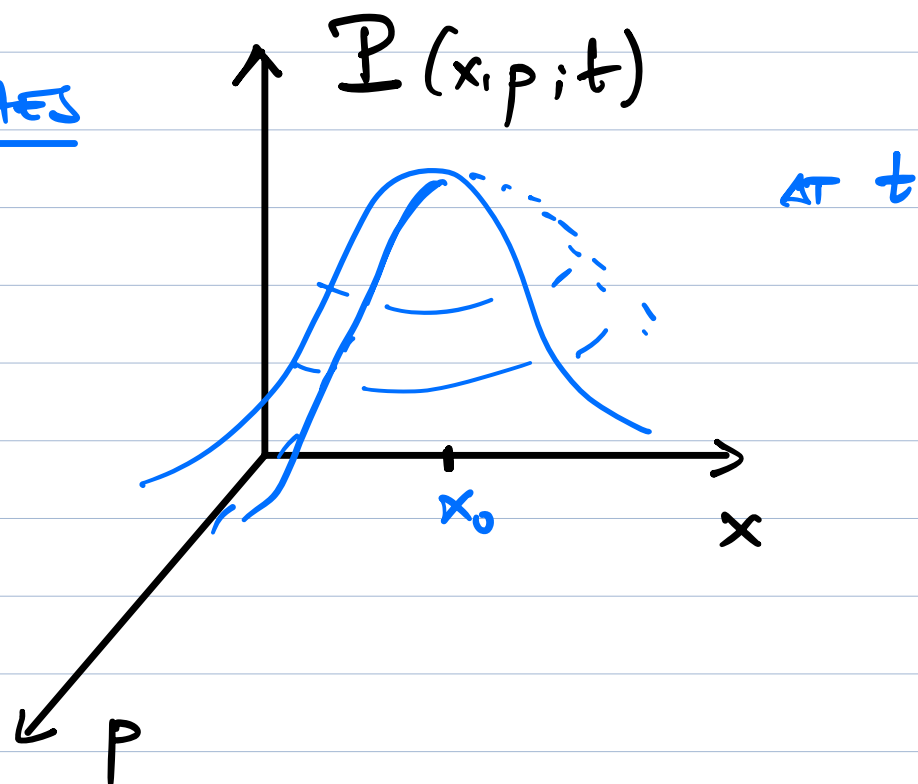
PROB. OF PARTICLES
HAVING $\{x, p\}$ AT
TIME t KNOWING
INITIAL PROB.

eg. JUST x (OVERDAMPED) $d=1$ $N=1$

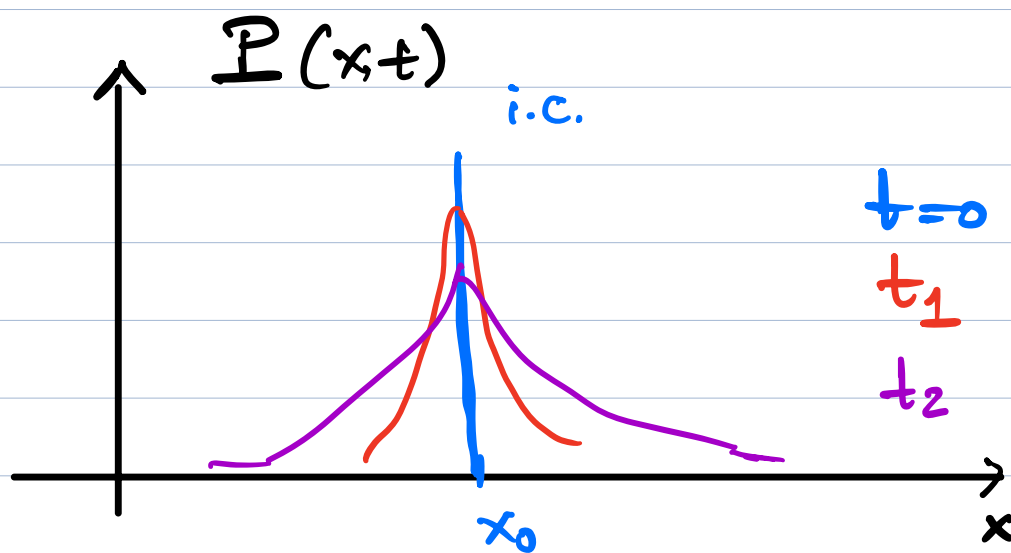


READ $P(x; t)$ HERE

SKETCHES



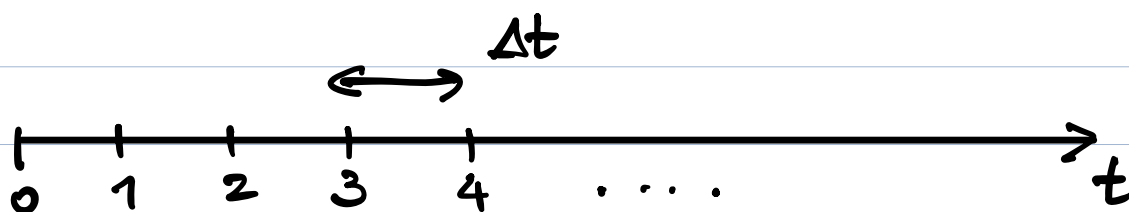
EASIER TO VISUALIZE
OVER-DAMPED LIMIT $t \gg m/\gamma_0$



HOW DO WE DERIVE AN EQ. THAT
RULES THE t -EVOL OF

$$P(p, x; t) \quad ?$$

THINK ABOUT THE DISCRETIZED IN
TIME PROCEJS



$$t_k = k \cdot \Delta t$$

$$k = 0, 1, 2, \dots$$

FOR A GENERIC NON-MARKOV PROBLEMS

$$P(p', x, t + \Delta t) = \int dx' \int dp' P(p, x; t + \Delta t | p', x'; t; \text{WHOLE HISTORY BEFORE}) \\ P(p', x'; t; \text{WHOLE HISTORY BEFORE})$$

BUT CAN'T DO MUCH WITH THIS

MARKOV ASSUMPTION

ONLY FOR δ -CORR NOISE !

$$P(p, x; t + \Delta t) = \int dp_0 \int dx_0 P(p, x; t + \Delta t | p_0, x_0; t) P(p_0, x_0; t)$$


IT'S LIKE AN I.C.

THAT'S WHY I CALL IT x_0 FROM NOW ON

TRANSITION PROB.

$$\mathbb{P}(p, x; t + \Delta t \mid p_0, x_0; t) = \langle \delta(p - p(t + \Delta t)) \delta(x - x(t + \Delta t)) \rangle$$

IMPOSING

$p = p(t + \Delta t)$ $x = x(t + \Delta t)$

 GIVEN BY A TRAJECTORY

WITH THE i. c. $\begin{cases} p(t) = p_0 \\ x(t) = x_0 \end{cases}$

AND THE EVOLUTION GIVEN BY THE
LANG EQ. WITH NOISE $\xi(t)$
AT TIME t

FOCUS ON ADDITIVE NOISE CASE.

GENER. TO MULT NOISE IS
POSSIBLE

TAKE OVER DAMPED LIMIT (EASIER)
GEN TO UNDER-DAMPED ALSO OK

THE UPDATING EQ FOR $\mathcal{P}(x, t)$ IS:

$$\mathcal{P}(x; t + \Delta t) =$$

$$\int dx_0 \mathcal{P}(x, t + \Delta t | x_0, t) \mathcal{P}(x_0, t)$$

WITH

$$\mathcal{P}(x; t + \Delta t | x_0, t) = \langle \delta(x - x(t + \Delta t)) \rangle$$

LET'S LOOK AT THE RHS 

USE THE LANGEVIN EQ.

$$\gamma \dot{x}(t) = F[x(t)] + \xi(t)$$

AND DISCRETIZE IT:

$$x(t+\Delta t) = x(t) + \Delta x = x_0 + \Delta x$$

WITH (ITO DISCRETIZATION)

$$\Delta x = \frac{1}{\gamma} F[x_0] \Delta t + \frac{\Delta t}{\gamma} \xi(t)$$

TAYLOR EXPAND THE δ :

$$\delta(x - x(t+\Delta t)) = \delta(x - x_0 - \Delta x)$$

$$\approx \delta(x - x_0) - \delta'(x - x_0) \Delta x$$

$$+ \frac{1}{2} \delta''(x - x_0) \underbrace{\Delta x^2}_{\text{IMPORTANT!}} + \mathcal{O}(\Delta x^3)$$

AND TAKE THE AVER. OVER NOISE

$$\langle \delta(x - x(t + \Delta t)) \rangle \simeq$$

$$\simeq \delta(x - x_0) - \delta'(x - x_0) \langle \Delta x \rangle$$

$$+ \frac{1}{2} \delta''(x - x_0) \langle \Delta x^2 \rangle$$

WITH

$$\langle \Delta x \rangle = \frac{1}{\gamma} F[x_0] \Delta t + \frac{\Delta t}{\gamma} \langle \dot{x}(t) \rangle$$

$$\langle \Delta x^2 \rangle = \frac{1}{\gamma^2} F^2[x_0] \Delta t^2$$

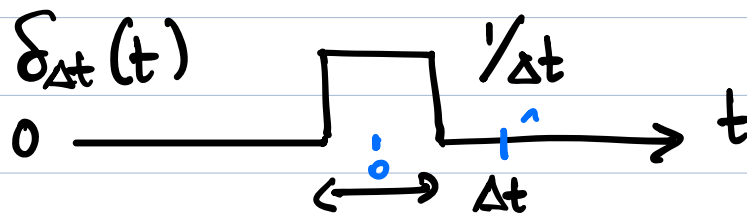
$$+ \frac{2}{\gamma^2} F[x_0] \Delta t^2 \langle \dot{x}(t) \rangle$$

$$O(\Delta t) \leftarrow + \frac{\Delta t^2}{\gamma^2} \underbrace{\langle \dot{x}^2(t) \rangle}_{\frac{2\gamma^2 k_B T}{\Delta t}}$$

why $\langle S^2(t) \rangle \propto 1/\Delta t$?

DISCRETIZATION of

$$\langle S(t) S(t') \rangle = 2\gamma h_B T \delta(t-t')$$



LIKE THIS $\delta_{\Delta t}(t) \propto \frac{1}{\Delta t} \xrightarrow{\Delta t \rightarrow 0} \infty$

$$\int_{-\infty}^{\infty} dt \delta_{\Delta t}(t) = \frac{1}{\Delta t} \cdot \Delta t = 1$$

PROPS of δ : ok

All in all

NEGLIGIBLE

$$\langle \Delta x^2 \rangle = \frac{1}{\gamma^2} F[x_0] \Delta t^2 + \frac{\Delta t^2}{\gamma^2} 2\gamma k_B T \frac{1}{\Delta t} + \dots$$

also

$$= 2 \Delta t \frac{k_B T}{\gamma} + \mathcal{O}(\Delta t^2)$$

NOW, BACK IN THE TRANSITION PROB

$$\mathbb{P}(x; t + \Delta t \mid x_0; t) = \langle \delta(x - x(t + \Delta t)) \rangle$$

$$= \delta(x - x_0) - \delta'(x - x_0) \langle \Delta x \rangle$$

$$+ \frac{1}{2} \delta''(x - x_0) \langle \Delta x^2 \rangle$$

$$= \delta(x-x_0) - \delta'(x-x_0) \frac{1}{\gamma} F(x_0) \Delta t \\ + \frac{1}{2} \delta''(x-x_0) \frac{2 \Delta t \frac{\hbar \gamma T}{\gamma}}{\gamma} + \mathcal{O}(\Delta t^2)$$

AND INSERTING IN THE UPDATING EQ
FOR $\mathcal{I}$ (KEEPING $\mathcal{O}(\Delta t)$)

$$\mathcal{I}(x, t+\Delta t) = \int dx_0 \underbrace{\mathcal{I}(x, t+\Delta t | x_0, t)}_{\text{REPLACE}} \mathcal{I}(x_0, t)$$

$$= \int dx_0 \left[\delta(x-x_0) - \delta'(x-x_0) \frac{F(x_0)}{\gamma} \Delta t \right. \\ \left. + \delta''(x-x_0) \frac{\hbar \gamma T}{\gamma} \Delta t \right] \mathcal{I}(x_0, t)$$

INTEGRATE

$$= \mathcal{I}(x, t) - \frac{1}{\gamma} \frac{\partial}{\partial x} (F(x) \mathcal{I}(x, t)) \Delta t \\ + \hbar \gamma T / \gamma \frac{\partial^2 \mathcal{I}(x, t)}{\partial x^2} \Delta t$$

ARRANGE ON LEFT-HAND-SIDE

$$\gamma \frac{P(x, t + \Delta t) - P(x, t)}{\Delta t} \rightarrow \gamma \frac{\partial P(x, t)}{\partial t}$$

$$= - \frac{\partial}{\partial x} (F(x) P(x, t)) + k_B T \frac{\partial^2 P(x, t)}{\partial x^2}$$

FOKKER-PLANCK EQ.

IN PHASE SPACE

KRAMERS EQ

BOTH MOMENTUM & POSITION

UNDERDAMPED

$$\frac{\partial P}{\partial t} = \underbrace{- \frac{p}{m} \frac{\partial P}{\partial x} + \frac{\partial}{\partial p} (-F P)}_{\text{HAMILT. FLOW}}$$

$$+ \underbrace{\gamma \frac{\partial}{\partial p} (p P) + \gamma k_B T \frac{\partial^2 P}{\partial p^2}}_{\text{DISSIPATIVE PART}}$$

DISSIPATIVE PART

BOTH CASES

ONLY
FOR
MARKOV!

IF COLOURED NOISE $\Rightarrow$
NO LOCAL - IN TIME
DIFF EQ LIKE THESE
CAN BE DERIVED.

STILL USEFUL SINCE MANY CASES
HAVE WHITE - UNCORRELATED NOISE
NO MEMORY

EXTENSION TO MULT. NOISE $\Rightarrow$ OK.
BUT DISCRETIZATION HAS TO BE DONE CAREFULLY

WHAT IS IT USEFUL FOR?

SOME CALC. ARE EASIER &

PROOFS OF APPROACH TO EQUIL

GIBBS-BOLTZMANN

$$F = -V'(x) \Rightarrow$$

$$P(x_p) = \frac{e^{-\beta H_{\text{sys}}(x_p)}}{Z(\beta)}$$

$$H_{\text{sys}} = K(p) + V(x)$$

IS A STATIONARY SOLUTION $\frac{\partial P(x_p; t)}{\partial t} = 0$

BUT IF WE DON'T HAVE KRAMERS / FP?

IT'S HARDER TO PROVE THAT THE
LANGEVIN PROCESS TENDS TO GIBBS-BOLTZ.

IF CONFINING POT $V(x)$ AND
NOISE IN EQUIL $\Rightarrow$ IT DOES, EVEN FOR
COLOURED NOISE

IF MULT. WHITE NOISE $\Rightarrow$ SOME ISSUES.

t_{eq} FINITE BUT COULD BE $t_{\text{eq}}(N)$

LINDBLAD EQ - QUANTUM

TAKE A QUANTUM SYSTEM IN A MIXED STATE

$$\hat{\rho} = \sum_{\alpha} p_{\alpha} |\psi_{\alpha}\rangle \langle \psi_{\alpha}|$$

↗ BASIS OF HILBERT SPACE

$$\sum_{\alpha} p_{\alpha} = 1$$

IT'S AN OPERATOR $\Rightarrow$ FOLLOWS HEISENBERG

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)]$$

IF SYST IS ISOLATED FROM REST OF THE WORLD

TAKE SYST + ENV + COUPLING

$$\hat{H} = \hat{H}_{\text{SYST}} + \hat{H}_{\text{BATH}} + \hat{H}_{\text{COUPLING}}$$

COUPLES SYST'S OPERATOR, SAY $\hat{x}$ TO
BATH OPERATORS, SAY $\{b_a^\dagger\}$

INTEGRATE AWAY THE $\{b_a^\dagger\}$ 'S $\Rightarrow$ REDUCE

DEFINE $\rho_{\text{SYST}}^{\text{RED}} = \text{Tr}_{\text{BATH}} \rho$
 $\{b_a^\dagger\}$

INTEGRATE AWAY OPERATORS FROM BATH.

$\Rightarrow$ UNDER SIMILAR ASSUMPTIONS TO
THE ONE OF LANGEVIN EQ'S DERIV

- ASSUME COUPLING SWITCHED ON AT $t=0$
- ASSUME BATH MUCH FASTER
AND MUCH LARGER $N_b \gg 1$ THAN SYST
- MARROU APPROX.

DERIVE LINDBLAD'S EQ:

$$\frac{d\hat{\rho}_{\text{sys}}^{\text{REF}}}{dt} = \frac{-i}{\hbar} [\hat{H}_{\text{sys}}, \hat{\rho}_{\text{sys}}^{\text{REF}}]$$

$$+ \sum_{mn} h_{mn} \hat{L}_m^{\dagger} \hat{\rho}_{\text{sys}}^{\text{REF}} \hat{L}_m$$

$$- \frac{1}{2} \{ \hat{L}_m^{\dagger} \hat{L}_m, \hat{\rho}_{\text{sys}}^{\text{REF}} \}$$

FROM THE COURSE TO DATA.

IN PRACTICE: CHOOSE $\{ \hat{L}_m \}$

WEISS 2000, BAUER PETRUCIANI

SCHIRO' & FAZIO SciPost 2025

J. MARINO ET AL 2025

PROBLEMS

- IT DOES NOT HAVE $\frac{e^{-\beta \hat{H}_{\text{sys}}}}{Z}$ AS ASYMPT. STAT. SOLUTION!

- IT'S LOCAL IN TIME $\Rightarrow$ NO MEMORY
SIMILAR TO KRAMERS / FOKKER-PLANCK

SOL. STILL $\Rightarrow$ OK FOR Q OPTICS, ETC.
AND CAN GIVE SOME IDEAS

$\Rightarrow$ MY SEMINAR ON QUANTUM ACTIVE

FUNCTIONAL FORMALISM

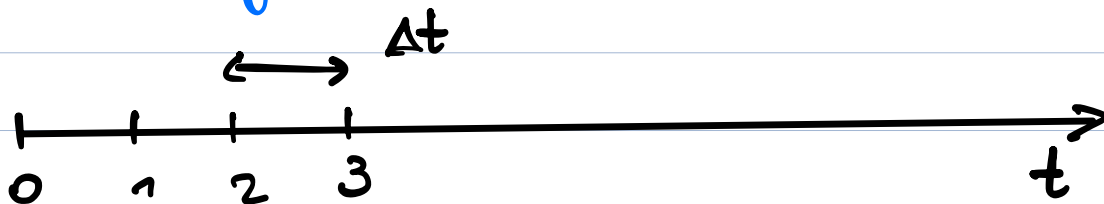
MSR-FD

CLASSICAL

$$\langle A(x(t)) \rangle_{\xi}$$

WE WANT TO CALC.
SUCH AVER. W/ $A(x)$
A FUNCTION - OBSERV.
OF x

SIMPLIFY AND USE OVERDAMPED



$$t_0 = l \cdot \Delta t$$

$$x_{\xi}(t)$$

SOLUTION TO LANG. EQ.
STRESS ξ DEP.

$$\text{SO } \langle A(x_{\xi}(t)) \rangle_{\xi}$$

SKETCH OF CONSTRUCTION

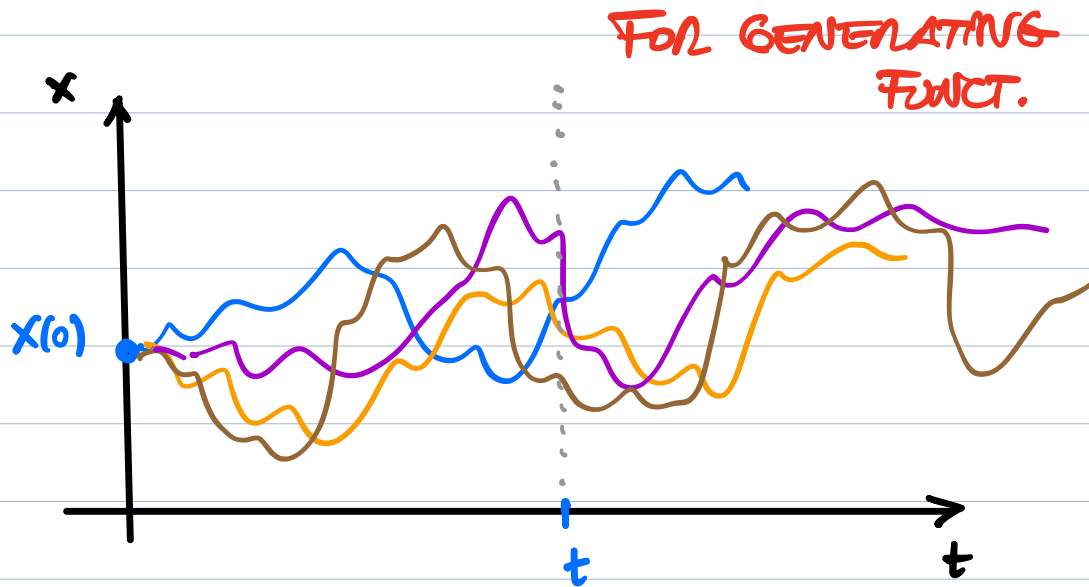
$$\langle A(x_\xi(t)) \rangle_\xi =$$

$$= \int \mathcal{D}\xi \, P(\xi|t) \, A(x_\xi(t))$$

AVERAGE OVER ALL NOISE
REALIZATIONS WITH THEIR
WEIGHT

$$\int \underbrace{\pi d\xi(t_\ell)}_{\text{DISCRETIZED INT MEASURE}} \underbrace{e^{-\frac{1}{k_B T} \sum_{\ell\ell'} \xi_\ell \Gamma_{\ell\ell'}^{-1} \xi_{\ell'}}}_{\text{DISCRETIZED GAUSSIAN WEIGHT}}$$

FOR COLOURED NOISE IT MIXES ξ_ℓ
AT DIFF TIMES THROUGH KERNEL



$$Z_{\text{dyn}}(\lambda) = \left\langle e^{\int_0^t dt' \lambda(t') x(t')} \right\rangle_{\xi}$$

MEANS A SUM OVER ALL REALIZATIONS OF ξ
WITH THEIR

$$P[\xi(t)]$$

AND $x_{\xi}(t)$ SOL. OF LANG EQ.

THE TRICK

WHICH I'LL NOT EXPLAIN FOR
LACK OF TIME

TRANSFORM THIS PATH INTEGRAL

INTO ONE WHERE ONE INTEGRATES
OVER

$x(t)$ AND $i\hat{x}(t)$

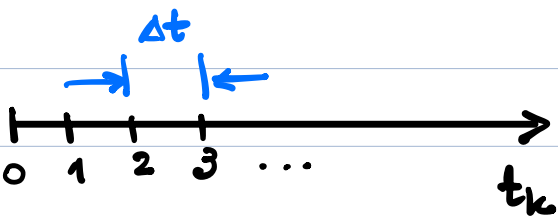
ORIGINAL VAR. AUXIL. VAR.

AND OBTAIN A GENERATING FUNCT.

$$\begin{aligned} Z_{\text{dyn}}[\lambda] &= \\ &= \int \mathcal{D}x \int \mathcal{D}i\hat{x} e^{-S[x, i\hat{x}]} \\ &\quad e^{\int dt \underbrace{\lambda(t) A(x(t))}_{\text{SOURCE}}} \end{aligned}$$

SOURCE

HINTS ON HOW TO OBTAIN THE ACTION S

- DISCRETIZE TIME
- 

- DISCRETIZE LANGEVIN EQ.

- IF INERTIA PRESENT OR COLOURED NOISE
≠ PRESCRIPTIONS ARE EQUIVALENT FOR
 $\Delta t \rightarrow 0$

- IF OVERDAMPED & WHITE NOISE :
CAREFUL !
SPECIALLY WHEN CONSTRUCTING THE GEN.
FUNCT.

- IMPOSE THE DISC. LANG. EQ $\forall \delta$

- EXPON. $\delta \sim$ FOURIER TRANSF.

- INTEGRATE OVER NOISE

THE INFLUENCE FUNCTIONAL

THE ZWANZIG-KAWASAKI WITHIN PATH INT.

$$\mathcal{Z}_{\text{dyn}}[\lambda] = \int \prod_a d q_a \prod_a d \pi_a \int dx \int dp \ e^{\int_0^t dt' \lambda(t') x(t')} \\ \exp \left\{ -S[\{x, p\}; q_a, \pi_a] \right\} \quad \lambda_{\text{SOURCE}}$$

INTEGRATE AWAY THE $\{ \pi_a, q_a \}$

$$\mathcal{Z}_{\text{dyn}}[0] = \int dx \int dp \ \exp \left\{ -S[\{x, p\}; \hbar\beta T, \Gamma] \right\}$$

$$S = S_{\text{sys}}[\{x, p\}] + \underbrace{S_{\text{INF}}[\{x, p\}; \hbar\beta T, \Gamma]}$$

BATH EFFECT $\Rightarrow$ INFLUENCE FUNCT.

THIS IS A FORMALISM WHICH CAN BE GENERALIZED
QUANTUM MECHANICALLY.

FEYNMAN - VERNOU 60s

FOR EXAMPLE, FOR A SINGLE VARIABLE BILINEARLY
COUPLED TO AN OSCILLATOR BATH

Classically

$$\begin{aligned}
 S[x, i\hat{x}] = & \underbrace{\ln P_0(x(0), \dot{x}(0))}_{\text{INITIAL COND.}} \\
 & - \int_0^t dt' i\hat{x}(t') \left\{ m\ddot{x}(t') + \underbrace{\int_0^{t'} dt'' \Gamma(t'-t'') \dot{x}(t'') - F[x(t')]}_{\text{LANGE EQ. (APART FROM NOISE)}} \right\} \\
 & + \underbrace{\frac{k_B T}{2} \int_0^t dt' \int_0^t dt'' i\hat{x}(t') \Gamma(t'-t'') i\hat{x}(t'')}_{\text{INTEGRATION OVER NOISE}} \\
 & + \underbrace{\ln J[x]}_{\text{JACOBIAN - HAS TO DO WITH DISCRETIZATION}}
 \end{aligned}$$

USED TO DEFINE THE STOCH PROCESS AND
BUILD THE PATH INTEGRAL

WHY IS THIS FORMALISM USEFUL? EXACT NO APPROX.!

- PERTURB. TH.
- SADDLE-POINT / MEAN-FIELD eg.
- SYMMETRIES $\rightarrow$ WARD IDENTITIES, FUNCT THS.

CORRELATIONS

$$\langle x(t) x(t') \rangle = \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S[\]} x(t) x(t')$$

LINEAR RESPONSE

$$\left. \frac{\delta \langle x(t) \rangle_h}{\delta h(t')} \right|_{h=0} = \frac{\delta}{\delta h(t')} \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S[\]} x(t) \Big|_{h=0}$$

$\uparrow$ in:

$$F_h[x(t)] = F[x(t)] + h(t) x(t)$$

RECALL $e^{\int_0^t dt'' i \hat{x}(t'') F_h[x(t'')]} \quad \text{IN PATH INTEG.}$

$$\Rightarrow \left. \frac{\delta \langle x(t) \rangle_h}{\delta h(t')} \right|_{h=0} = \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S[\]} x(t) i \hat{x}(t')$$

$$= \langle x(t) i \hat{x}(t') \rangle$$

CAUSALITY
 $\neq 0 \quad t \geq t'$

RESPONSE FIELD 52

SYMMETRIES

PROVE FDT IF

$P_0(x(0), \dot{x}(0))$ MAXWELL-BOLTZM.

$F[x] = -V'[x]$ POTENTIAL FORCE

WITH THE SAME

$V[x]$ USED TO DRAW i.c.

$\Gamma(t-t')$ THE SAME IN FRICTION AND NOISE CORR.

PROOF. IT'S BASED ON THE INVARI. OF ACTION
UNDER SIMULT TRANSF OF $\{x(t), \dot{x}(t)\}$

$$S[x(t), \dot{x}(t)] = S[x(-t), \dot{x}(-t)] + \underbrace{\beta \frac{d}{dt} x(-t)}_{\text{TRANSL.}}$$

↑ ↑
TIME REVERSAL TRANSL.

ALSO THE MEASURE $\mathcal{D}x \mathcal{D}\dot{x}$ IS INVARIANT
Σ INTEGRATION INTERVAL TOO

CONSEQUENCE ON
CORRELATIONS & LINEAR RESP.²

NEWTON INFLUENCE

NOTE THAT $S_{\text{sys}} + JAC$ & S_{diss} ARE SEPARATELY INVARIANT

$$\langle x(-t) | \hat{x}(-t') \rangle =$$



ALL THE $\langle \dots \rangle$ OPERATION IS INV. SINCE S & MEASURE ARE

$$\langle x(t) \left[\hat{x}(t') - \beta \frac{d}{dt'} x(t') \right] \rangle \Rightarrow$$

$$\langle x(t) | \hat{x}(t') \rangle = \langle x(-t) | \hat{x}(-t') \rangle + \beta \langle x(t) | \dot{x}(t') \rangle$$

FDT

$$R(t, t') = R(-t, -t') + \beta \frac{\partial}{\partial t'} C(t, t')$$

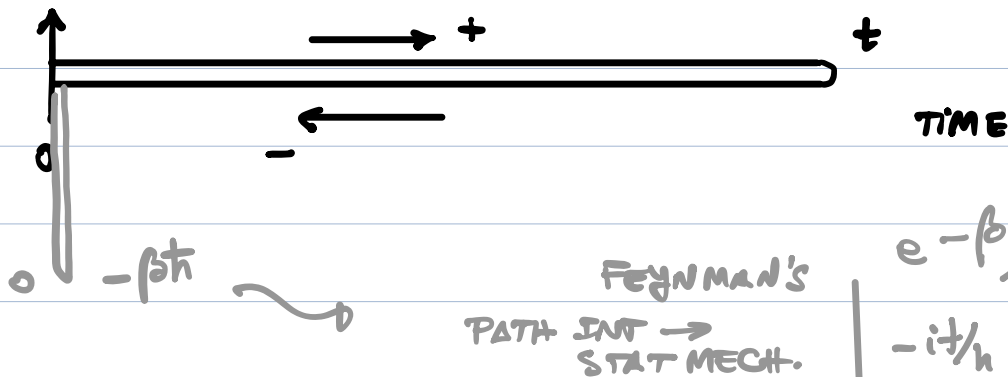
THEREFORE, IF FDT $\Rightarrow$ EQUIL. A LA
MAXWELL-BOLTZMANN

THEN THESE PROCESSES DO TAKE SYST
TO EQUIL. UNDER ABOVE CONDITIONS

QUANTUM

VERY SIMILAR !

COMPLEX PLANE



$$e^{-\beta \hat{H}} / \mathcal{Z} = \frac{1}{\mathcal{Z}} \int \mathcal{D}\vec{x}$$

$$-i\hbar \rightarrow -\beta$$

$$t = -i\tau$$

$$\tau = \beta\hbar$$

IN ORDER TO CALCULATE CORREL. LIKE

$$\langle \hat{A}(t) \hat{B}(t') + \hat{B}(t') \hat{A}(t) \rangle$$

NO TIME ORDERED, TRICK GO FORTH & BACK.

$$\frac{x^+(t) + x^-(t)}{2} \xrightarrow{\hbar \rightarrow 0} x(t)$$

$$\frac{x^+(t) - x^-(t)}{\hbar} \xrightarrow{\hbar \rightarrow 0} i\vec{x}(t)$$

BATH:

$$\tilde{\Gamma}(t-t') = \int_0^{\infty} d\omega S(\omega) \coth\left(\frac{\beta\hbar\omega}{2}\right) \cos \omega(t-t')$$

$$\coth x : \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{2}{\cancel{1+x} - \cancel{1+x}} = \frac{1}{x}$$

$$\lim_{\beta\hbar \rightarrow 0} \tilde{\Gamma} = \int_0^{\infty} d\omega S(\omega) \frac{2}{\beta\hbar\omega} \cos \omega(t-t')$$

$$= \frac{k_B T}{\hbar} \underbrace{\int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{\omega} \cos \omega(t-t')}_{\Gamma(t-t')}$$

THIS KERNEL APPEARS AS A TERM IN $e^{\frac{i}{\hbar} S_{\text{diss}}}$

$$\overset{\text{NOISE}}{\frac{i S_{\text{diss}}}{\hbar}} = \frac{(-1)}{\hbar} \int_0^t dt' \int_0^t dt'' \underbrace{(x^+(t') - x^-(t'))}_{i\hat{x}(t')\hbar} \tilde{\Gamma}(t'-t'') \underbrace{(x^+(t'') - x^-(t''))}_{i\hat{x}(t'')\hbar}$$

$$\overset{\text{CLASSICAL}}{\hbar \rightarrow 0} \rightarrow \frac{-\cancel{\hbar^2}}{\cancel{\hbar}} \frac{k_B T}{\cancel{\hbar}} \int_0^t dt' \int_0^t dt'' i\hat{x}(t') \Gamma(t-t') i\hat{x}(t'')$$

THERE IS ANOTHER TERM THAT DOES NOT DEP. ON TEMP. AND PLAYS THE RÔLE OF THE RETARDED FRICTION (RESPONSE OF BATH) RELATED BY QUANTUM FDT TO THIS ONE

$$\begin{aligned}\gamma'(\omega) &= -\Theta(t-t') \int_0^\infty d\omega S(\omega) \sin \omega (t-t') \\ &= \frac{1}{4\hbar} \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega'}{\pi} \frac{1}{\omega - \omega' + i\epsilon} \hbar \frac{\beta \hbar \omega'}{2} 2\hbar \tilde{\Gamma}(\omega')\end{aligned}$$

APPLICATIONS

— NOISE-INDUCED PH. TRANS

HOT TOPIC IN QUANTUM

SCHWINGER - KELDYSH
+ FEYNMAN VERNON

MSR-JD

COMPLETE FORMALISM THAT LET'S US TREAT
ANY KIND OF BATH - NOT ONLY OHMIC

SYMMETRIES

TRANSFORMATION OF PATHS IN THE
COMPLEX PLANE

C. ALON, G. BIRLOU & LFC 18

SUMMARY 1st & 2nd LECTURES

- I JUSTIFIED THE GENERALIZED LANGEVIN EQ.
BEYOND BROWNIAN MOTION

GENERIC MEMORY KERNEL & NOISE-CORR
RELATED BY FDT IF EQ BATH

- SKETCHED

THE CLASSICAL GEN. FUNCT. CONSTRUCTION

- DISCUSSED ITS TIME-REV. SYMM. $\Rightarrow$ FDT
& THEN EQUIL. I HAVEN'T DONE IT
SEE NOTES IF INTER.

- SKETCHED THE CONSTRUCT OF QUANTUM
GEN FCT - SCHWINGER-KELDysh
w/ GENERIC BATH - FEYNMAN-VERNON
NOT DONE EITHER

- THE KERNELS FOR A QUANTUM
OTC BATHS CAN BE CALCULATED AS
WELL

IMPORTANT

$F = -V'$ AND
ALL THESE FORMALISMS SHOULD ADMIT (IF BATH IN EQ.)
TO TAKE SYST FOR $t \gg t_{eq}$ (PARAM.) TO CANON. EQUIL

$$\frac{e^{-\beta H_{\text{sys}}}}{\mathcal{Z}(\beta)}$$

NOT EASY IN GENERAL

→ OK FOR FOKKER PLANCK / KRAMERS

→ SYMM. IN $\mathcal{Z}_{dyn}$ FOR COLOURED NOISE

LINDBLAD ... UHM ...

$t_{eq} ??$

COULD BE VERY LONG AND SYST SIZE-
DEPENDENT

⇒ LONG OUT OF EQ. RELAXATION

STILL FOR EQ. BATH $\mathcal{Z}$ $F = -V'$

LET'S DISCUSS SOME PHYSICAL PROBLEMS NOW.

1- SEP. OF TIME SCALES p vs. $x \Rightarrow$ ex.

2- DROP INERTIA, JUST x . SEPARATION OF TIME SCALES IN ITS OWN BEHAVIOUR.

NOT SO SURPRISING. THINK ABOUT RW AGAIN, FOR

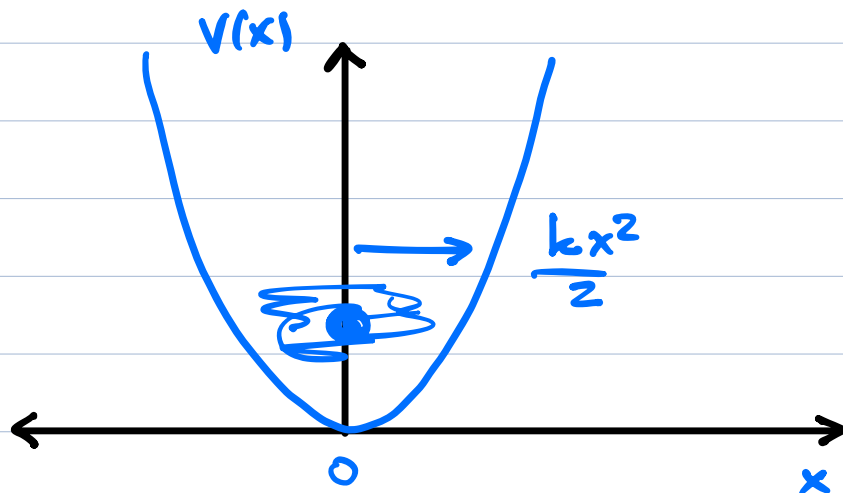
$t < t_{eq}^{(p)} \Rightarrow$ BALLISTIC $x(t) \approx \frac{p}{m} t$

$t > t_{eq}^{(p)} \Rightarrow$ DIFFUSIVE WHEN NOISE CONTROLS IF $V=0$

LET'S LOOK AT ANOTHER SIMPLE EX IN THE OVERDAMPED LIMIT

$$\gamma \dot{x} = -kx + S + h \rightarrow \text{TO MEASURE LINEAR RESP.}$$

A PARTICLE IN A HARM WEL UNDER ADD. WHITE NOISE



$$x(t) = x_0 e^{-\frac{k}{\gamma} t} + \frac{1}{\gamma} \int_0^t dt'' e^{-\frac{k}{\gamma} (t-t'')} [S(t'') + A(t'')] \quad t_{eq}^{(x)} = \frac{\gamma}{k} \quad x_0 = 0$$

$$\langle x(t) \rangle_h = \frac{1}{\gamma} \int_0^t dt'' e^{-\frac{k}{\gamma} (t-t'')} h(t'')$$

LINEAR RESPONSE TO AN INSTANT KICK

$$R(t, t') = \left. \frac{\delta \langle x(t) \rangle_h}{\delta h(t')} \right|_{h=0} = \frac{1}{\gamma} e^{-\frac{k}{\gamma} (t-t')} \underbrace{\theta(t-t')}_{\text{CAUSAL}}$$

WE NOTE THAT

$$R(t, t') = \bar{R}(t - t')$$

ABUSE OF NOTATION $R(t - t')$

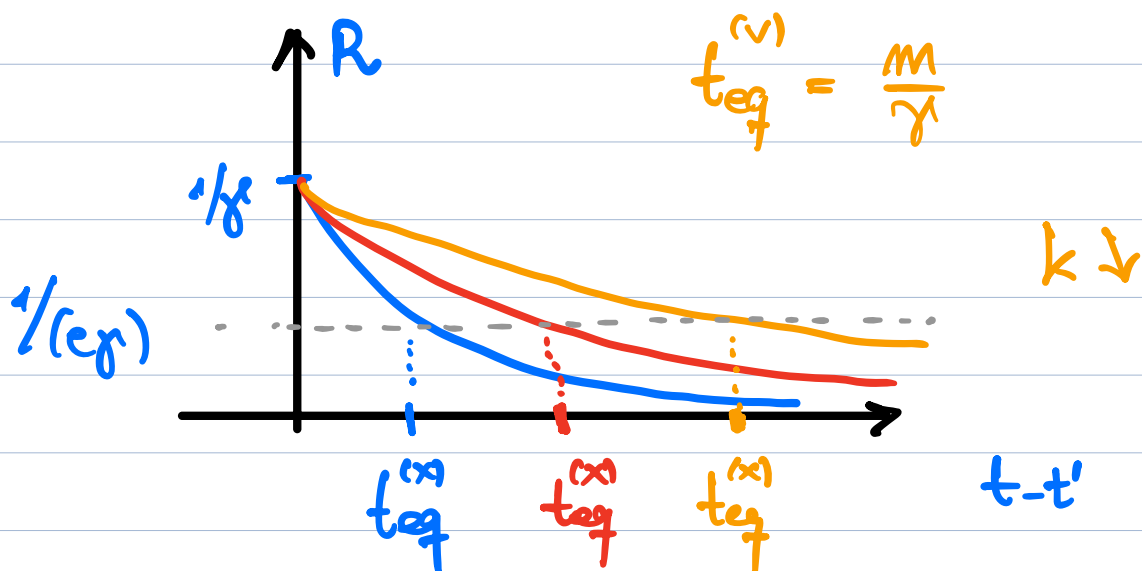
STATIONARY TIME-TRANSL. INVARIANT

$$t \rightarrow t + \underline{\Delta} \quad t' \rightarrow t' + \underline{\Delta}$$

$$\boxed{t_{eq}^{(x)} \equiv \frac{\gamma}{k}}$$

TIME SCALE FOR RELAX.

$$\text{NB } t_{eq}^{(x)} \xrightarrow{k \rightarrow 0} \infty$$



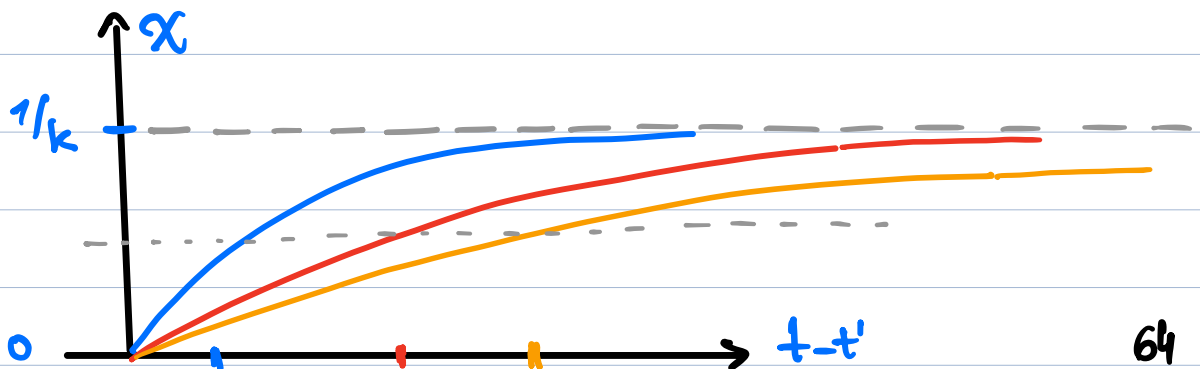
FOR DECREASING k THE $t_{eq}^{(x)}$
INCREASES

$$R(t-t') \xrightarrow[t \gg t_{eq}^{(x)}]{} 0$$

INTEGRATED LINEAR RESPONSE $\chi(t, t') = 0$

$$\begin{aligned}\chi(t, t') &= \int_{t'}^t dt'' R(t, t'') \\ &= \int_{t'}^t dt'' \frac{1}{\gamma} e^{-\frac{k}{\gamma}(t-t'')} \\ &= \frac{1}{\gamma} \frac{\gamma}{k} \left(1 - e^{-\frac{k}{\gamma}(t-t')} \right)\end{aligned}$$

$$\xrightarrow[t-t' \gg t_{eq}^{(x)}]{} \frac{1}{k}$$



SELF-CORRELATION

$$\langle x(t) x(t') \rangle_{h=0} = \frac{k_B T}{k} \left[e^{-\frac{k}{\gamma} |t-t'|} - \underbrace{e^{-\frac{k}{\gamma} (t+t')}}_{\substack{\rightarrow 0 \\ t+t' \rightarrow 0 \\ \text{iff } k \neq 0}} \right]$$

STATIONARY
SAME $t_{eq}^{(x)} = \frac{\gamma_0}{k}$

$$C(t, t') \xrightarrow{t' \gg \tau} \frac{k_B T}{k} e^{-\frac{k}{\gamma} |t-t'|} = C(t-t')$$

STATIONARY

TAKE A t' -DERIV.

$$\frac{\partial \langle x(t) x(t') \rangle}{\partial t'} \xrightarrow{t > t'} \frac{k_B T}{k} \frac{1}{\gamma} e^{-\frac{k}{\gamma} (t-t')} \quad t > t'$$

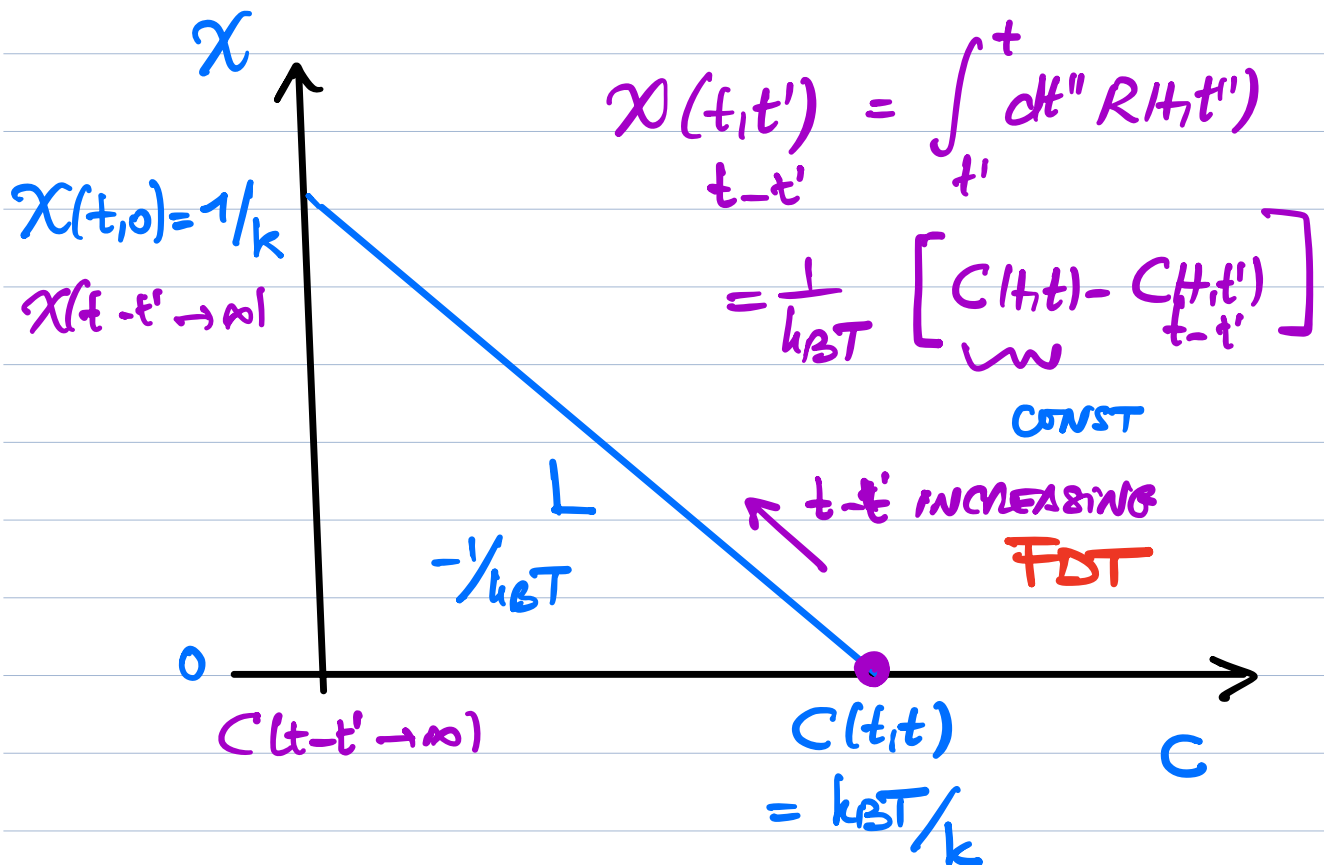
$$\partial_{t'} C(t, t') = \frac{k_B T}{\gamma} e^{-\frac{k}{\gamma} (t-t')}$$

cfr

$$R(t, t') = \frac{1}{k_B T} \partial_{t'} C(t, t') \quad \forall t > t'$$

FDT

$\forall t > t'$



FDT is ACTUALLY A MODEL INDEP

RESULT, VALID $\forall$ SYST EVOLVING IN EQUIL.

FOR ANY COLOURED NOISE

EXERCISE

STUDY WHAT HAPPENS IF YOU TAKE
 $k \rightarrow 0$

NO CONFINING POTENTIAL

RANDOM WALK $\rightarrow$ DIFFUSION

$$C(t) = \langle x^2(t) \rangle \rightarrow 2Dt$$

IS NOT A CONSTANT

$$\Rightarrow C(t) \neq C(t-t) = C(0)$$

NO STAT., NO FDT $\Rightarrow$ NO EQUIL

FLAT DIRECTIONS IN THE POTENTIAL
MAKE THE
EQUIL. TIME DIVERGE

WHAT HAPPENS IF YOU COUPLE THE SAME
PART IN A HARM WELL TO TWO $\neq$ BATH ?

CHOICE

$$\Gamma_f(t-t') = 2\gamma_f \delta(t-t')$$

τ_f

$\wedge$

$$\Gamma_s(t-t') = \frac{\gamma_s}{\tau_s} e^{-|t-t'|/\tau_s}$$

τ_s

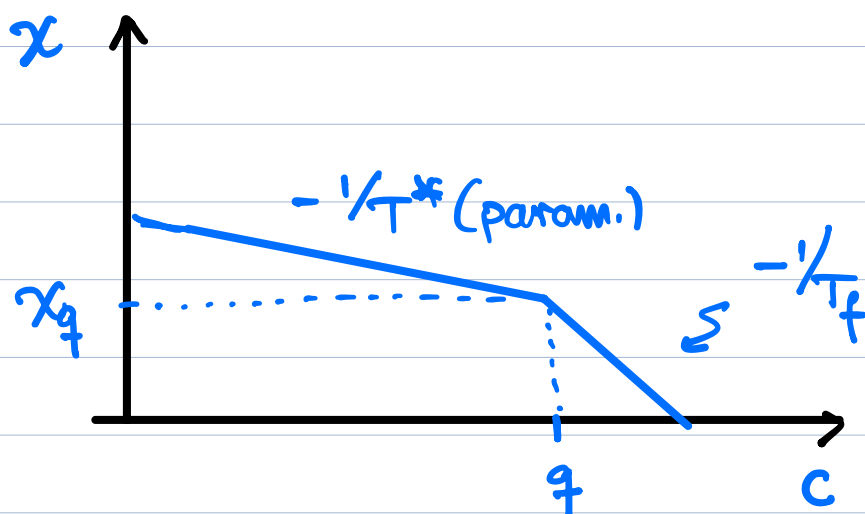
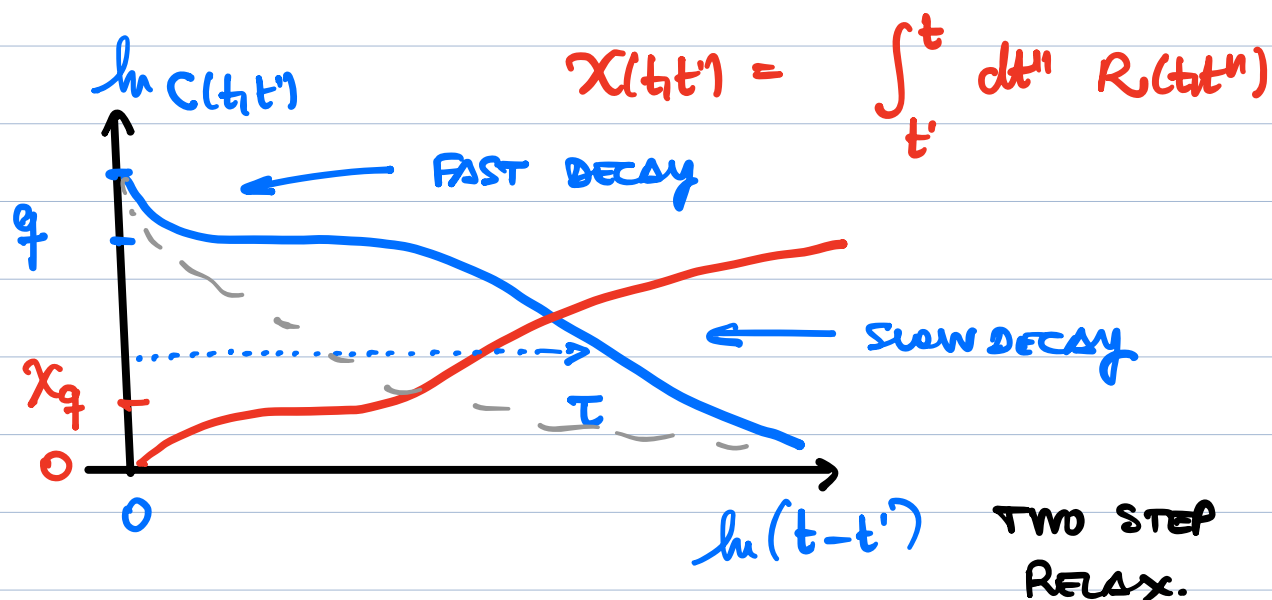
WE HAD $t_{eg}^{(x)} = \frac{\gamma}{k}$ BUT NOW IT'S CLEAR
THAT IT'LL BE MORE COMPLEX

$$\int_0^t dt' \left[\frac{\gamma_s}{\tau_s} (t-t') + 2\gamma_f \delta(t-t') \right] \dot{x}(t') =$$

$$= -kx(t) + S(t) + \eta(t) + h(t)$$

STILL A LINEAR 1ST ORDER DIFF EQ.
SOLVABLE ANALYTICALLY.

CHOOSE PARAM TO GET SEP. OF TIME SCALES

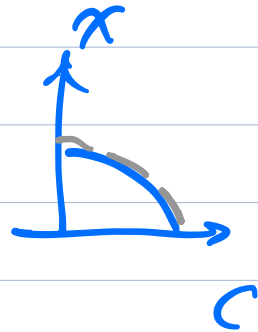


WITH TWO
TEMP
IN FOT
MEASUREMENT.

IMAGINE A SYST SUCH THAT

$$\lim_{t, t' \gg t^*} \chi(t, t') = \chi(c)$$

$$C(t, t') = c$$



ONE CAN DEFINE THE

EFFECTIVE TEMPERATURE

AS:

$$\frac{1}{k_B T_{\text{eff}}(c)} = \chi'(c)$$

- IN THE EQUIL. CASE $\chi(c) = \frac{1}{k_B T} (C(0) - c)$

$$T_{\text{eff}}(c) = T$$

- IN THE TWO SCALE CASE $\chi(c)$ PIECE WISE

$$T_{\text{eff}}(c) = \begin{cases} T_f & c > q \\ T_s & c < q \end{cases}$$

FINALLY COMPLEX SYSTEMS, BEYOND SINGLE VARIABLE.

DMFM WHAT IS IT ?

METHODS TO DEAL WITH DYNAMICS IN
MEAN-FIELD.

GROSSO MODO. SINGLE OUT ONE VARIABLE
OF YOUR WHOLE SYSTEM AND IDENTIFY
SELF-CONSISTENTLY WHICH IS THE
FRICTION & NOISE-NOISE KERNEL THAT
SHOULD GO IN THE GEN. LANG. EQ.

CALL x THE CHOSEN VARIABLE

$$m \ddot{x} + \underbrace{\gamma \dot{x}}_{\text{EXT BATH}} = \underbrace{\int_0^t dt' \mathcal{I}(t, t') x(t')}_{\text{INTERNAL EFFECT}} + \underbrace{\xi}_{\text{EXT BATH}} + \underbrace{\eta}_{\text{EXT BATH}}$$

$$\langle \xi(t) \xi(t') \rangle = 2\gamma k_B T \delta(t-t')$$

$$\langle \eta(t) \eta(t') \rangle = D(t-t')$$

NB Σ is like $\frac{d\Gamma(t-t')}{dt'}$ A "RESPONSE"

D is like $\Gamma(t-t')$ A "CORR"

WHO ARE Σ AND D ?

IN "MEAN-FIELD" MODELS THEY ARE FCTS OF
 $\{C, R\}$

$$C = \langle x(t) x(t') \rangle \quad \text{SELF-CORR}$$

$$R = \frac{\delta \langle x(t) \rangle}{\delta h(t')} \Big|_{h=0} \quad \text{LINEAR RESP.}$$

OF THE SYST ITSELF

eg. p -SPIN DISORDER MODEL $p \geq 3$

$$H_{\text{sys}} = - \sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} x_{i_1} \dots x_{i_p}$$

$$\sum_{i=1}^N x_i^2 = N \quad \text{SPH. CONST}$$

$J_{i_1 \dots i_p}$ QUENCHED RANDOM COUPLINGS
GAUSSIAN DIST CONVENIENTLY
NORMALIZED

$$\mathcal{D}(t, t') = J^2 \frac{p}{2} C^{p-1}(t, t')$$

$$\mathcal{I}(t, t') = J^2 \frac{p(p-1)}{2} C^{p-2}(t, t') R(t, t')$$

cfr J. KENT-BOBÍAS TALK

eg F. VAN WIJLAND'S LECTURES

OF COURSE, THIS IS A MUCH HARDER
PROBLEM TO SOLVE!

- SIMILAR STRUCT. CLASSICALLY & QUANTUM

- CAN BE PUT IN A COMPUTER

- CAN BE USED TO WRITE CLOSED EQS.

FOR $\{C, R\}$ $\rightarrow$ SCHWINGER-DYSON EQS.

BBGKY HIERARCHY CLOSED FOR MF MODELS

PHYSICS ? GLASSY MODELS.

THESE MODELS / EQS. HAVE A
DYN. PHASE TRANS.

- $T > T_d$

RELAX. AS IN THERMAL EQ.

$t > t_{eq} (N \rightarrow \infty)$: FINITE

STILL, SLOW RELAX. FOR $T \gtrsim T_d^+$

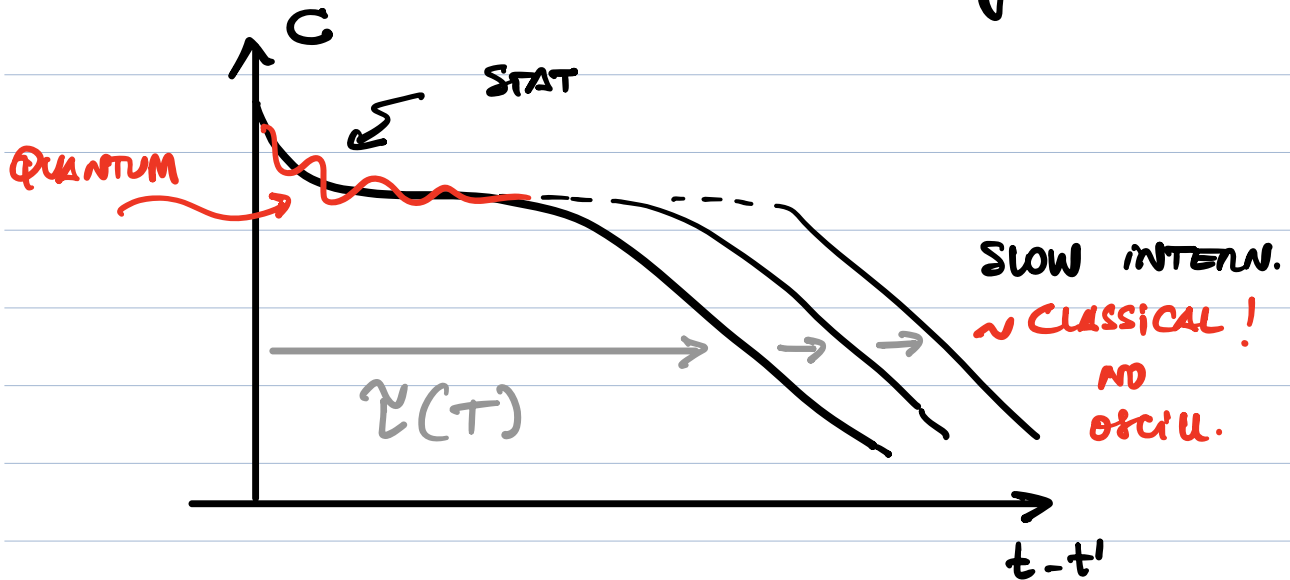
$\sim$ LIKE CRITICAL SLOWING DOWN
IN USUAL PH. TRANS.

FDT HOLDS.

TWO SCALE RELAX. $\Rightarrow$ SELF INDUCED.

$$T > T_d$$

FAST- CONTROLLED BY EXT BATH



$\chi(T) \uparrow$ FOR $T \downarrow$
TOWARDS T_d^+

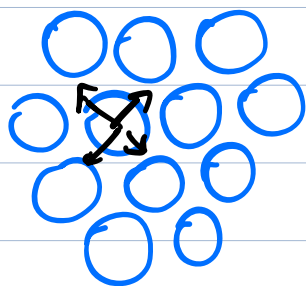
DIFFERENT CURVES FOR $\neq T$

$$\chi(c) = -\frac{1}{k_B T} (c(0) - c) \quad \text{FDT}$$

QUANTUM VERSION HOLDS TOO.

HOW DOES ONE UNDERSTAND THE
SEP. OF TIME SCALES?

NB C IS A GLOBAL CORR.



SHORT TIME
DIFF.

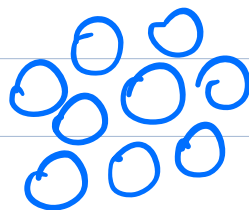
$$t - t' < \tau_c(\tau)$$

ONLY FLUCT.
WITHIN CAGES

LONG TIME - DIFF

$$t - t' > \tau_c(\tau)$$

STRUCT RELAX

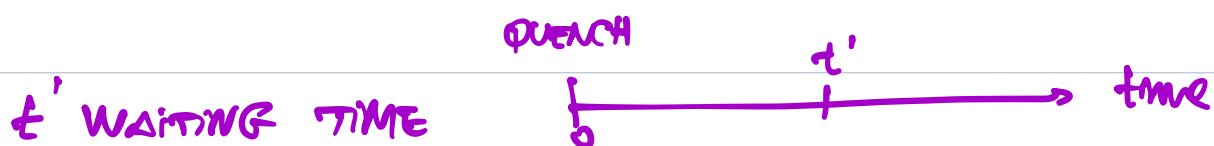


DIFF
ORIAN.
OF PART.

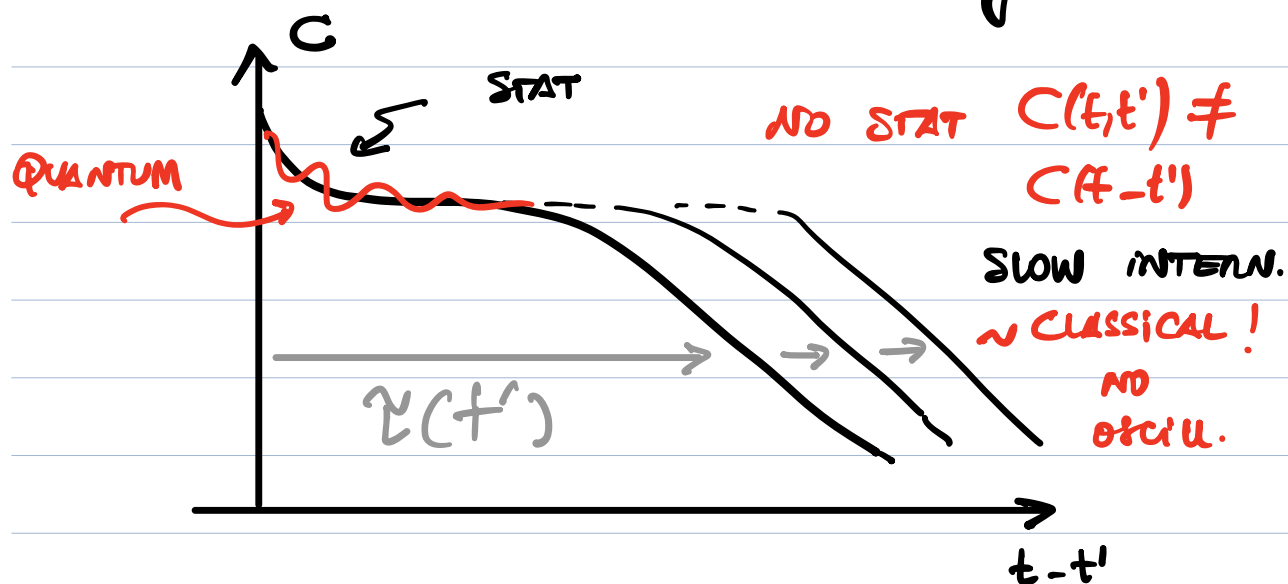
BUT ALL TAKES PLACE IN EQUIL.

- $T < T_d$

RELAX. OUT OF EQUILIBRIUM

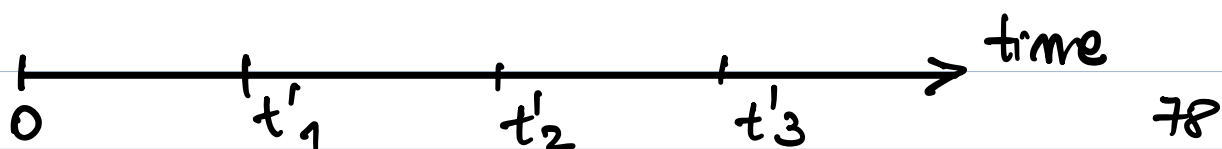


FAST-CONTROLLED BY EXT BATH



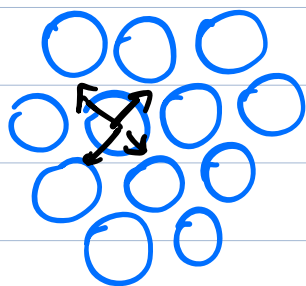
$$\gamma(t') \uparrow \text{ FOR } t' \uparrow$$

AGING



HOW DOES ONE UNDERSTAND THE
SEP. OF TIME SCALES?

NB C IS A GLOBAL CORR.



SHORT TIME
DIFF.

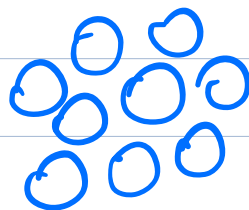
$$t - t' < \tau(t')$$

ONLY FUNCT.
WITHIN CAGES

LONG TIME - DIFF

$$t - t' > \tau(t')$$

STRUCT RELAX

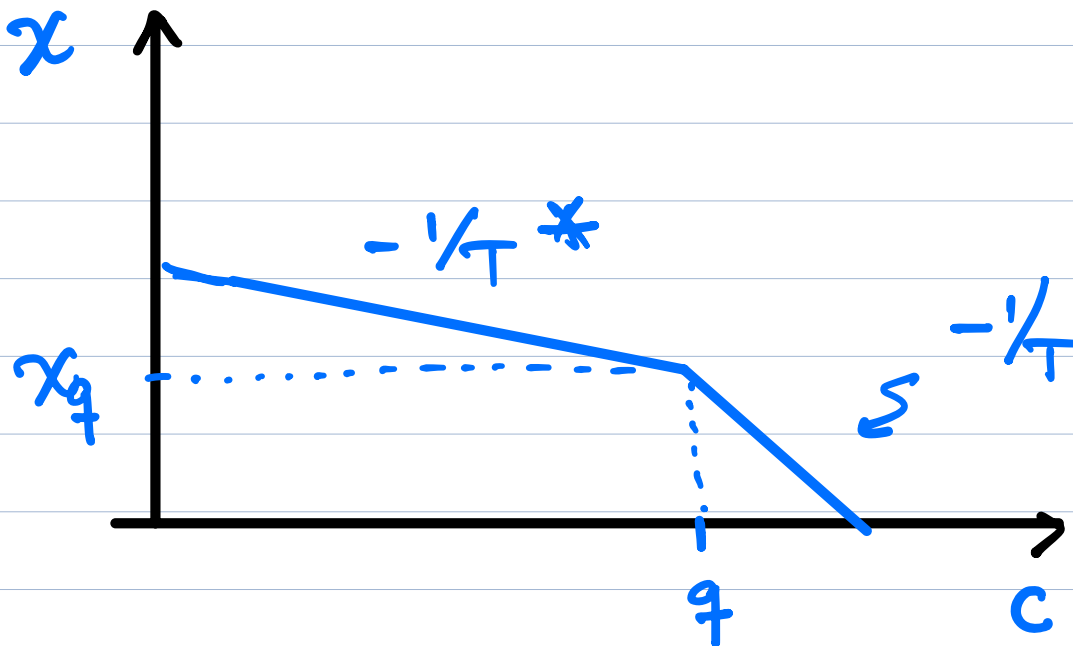


DIFF
ORGAN.
OF PART.

SECOND RELAX. OCCURS OUT OF EQ.

FWCT - DISS. RELATION

INTEGRATED REF. VS.
CONR. FCT.



WITH TWO
TEMP
IN FDT
MEASUREMENT.

Classical

$$-\frac{1}{T_{eff}(c)} = \frac{dX(c)}{dc} \begin{cases} c > q & = \frac{1}{T} \\ c < q & = \frac{1}{T^*} \end{cases}$$

THERE IS AN EFFECTIVE TEMP

$$T^* > T$$

SELF-GENERATED IN THE SYST

WHY $T^* > T$?

THIS IS A QUENCH FROM $T_i > T_d$
KIND OF "MEMORY" OF
THE INITIAL CONDITION

HEATING FROM $T_i < T_d$

ONE OBSERVER $T^* < T$
CONSISTENTLY
WITH THE IDEA OF "MEMORY"
FROM I.C.

RELATION TO PHASE SPACE STRUCT.

THERE IS !

$$\frac{1}{k_B T^*} = \frac{\partial \Sigma_{\text{conf}}}{\partial f_{\text{TAP}}} \quad \left| \begin{array}{l} \text{CONF ENTROPY} \\ \nearrow \text{COMPLEXITY} \\ \text{TAP} \\ f_{\text{threshold}} \end{array} \right.$$

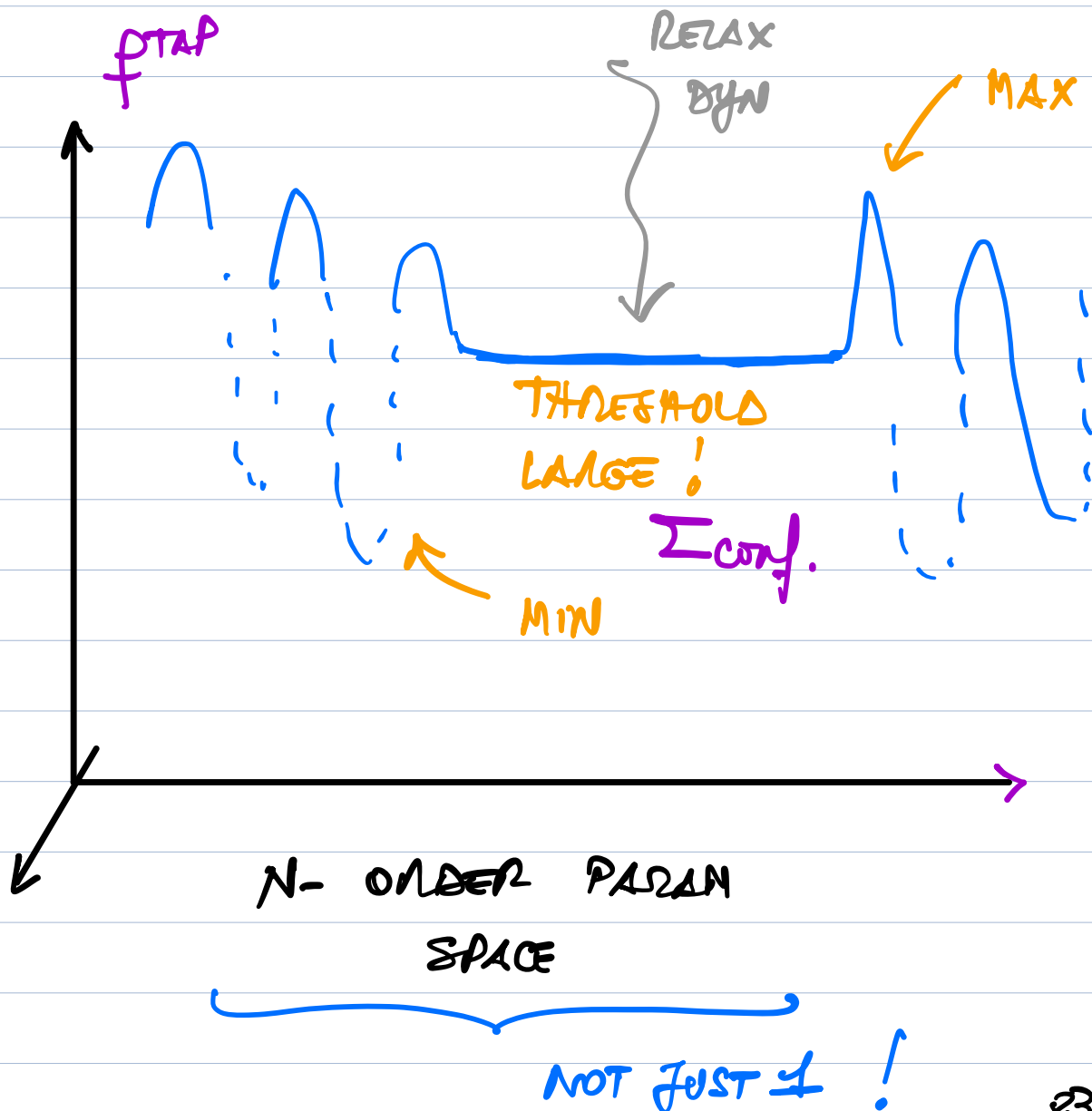
IN p-spin MODELS

cfr JALON'S TALK
BRUNO'S LECTURE

MICROCAN.
TEMP

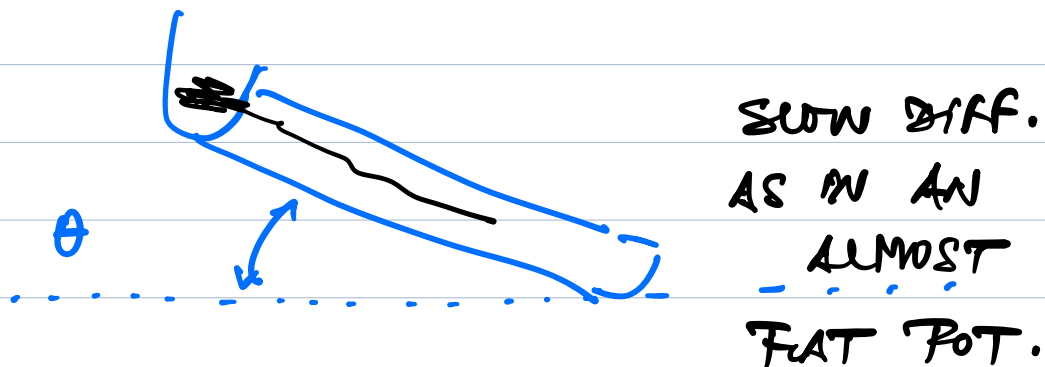
FREE-ENERGY LANDSCAPE PERSPECTIVE

$f^{TAP} \approx$ GINZBURG-LANDAU $T < T_d$
FREE-ENERGY DENS.



ONE HAS TO IMAGINE THE THRESHOLD
AS A NETWORK OF SUBTLY
TILTED CHANNELS

FAST RELAX.
AS IN A "HARMONIC WELL"



VERY SMALL θ

FREE-ENERGY LANDSCAPE
VIEW

QUANTUM

QUENCHES FROM DISORDERED PHASE
DOWN TO WITHIN THE ORDERED ONE

$C > q \Rightarrow$ QUANTUM FDT AT T

$C < q \Rightarrow$ CLASSICAL FDT AT T^*

Δ FREE-ENERGY LANDSCAPE
DESCRIPTION IS ALSO POSSIBLE

SUMMARY

1- INTRO TO STOCH PROCESSES

LANGVIN
FOKKER-PLANCK / KRAMERS
PATH INTEGRALS

2- A BIT OF PHENOM. OF GLASSY DYNAMICS.

SOME REFS

LFC ADV PHYS. 1st PART 2024

LFC ANNUAL REV. COND MATT 2024
LES MOUCHES 2002

ALSO NOTED IN MY WEB PAGE