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The Geometry of the Space of Selfadjoint Invertible Elements in a C*–algebra

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Let A be a C*-algebra with identity and G^s the set of all selfadjoint invertible elements of A. This paper is a study of the geometric properties of the manifold G^s. The action of the group G of invertible elements of A over G^s, given by $g \cdot a = (g^{-1})^* a g^{-1}$, defines Banach homogeneous spaces $G \to G^{s,a}$, where $G^{s,a}$ is the orbit of $a \in G^s$. It turns out that the $G^{s,a}$ are open and closed subsets of G^s and the principal bundles $G \to G^{s,a}$ carry natural connections. The horizontal lifting of (differentiable) curves γ in G^s are controlled by the differential equation $\dot{\Gamma} = -\frac{1}{2}\gamma\dot{\gamma}\Gamma$, which is called here *the transport equation* (an alternative approach based on multiplicative integrals is given in Section 8). Several G–bundles are studied, in particular the tangent bundle TG^s. One relevant point here is that the (left) polar decomposition $a = \nu\rho$ ($a \in G^s$, $\nu > 0$, ρ unitary) provides two structures: first it is easy to see that ρ is a reflection so that $\pi(a) = \rho$ defines a map $\pi : G^s \to P$ where P is the set of all $\rho \in A$ such that $\rho^* = \rho^{-1} = \rho$; second for a tangent vector $X \in T_aG^s$ the norm $\|X\|_a = \|\nu^{-1/2} X \nu^{-1/2}\|$ defines a Finsler structure on the bundle TG^s. This bundle carries a canonical connection determined by the transport equation, with covariant derivative defined by

$$D_X Y = X(Y) - \frac{1}{2}(Xa^{-1}Y + Ya^{-1}X)$$

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and parallel transport along a curve γ in G^s given by the transport function Γ of γ. Thus TG^s is endowed with the resulting structure of Finsler bundle with a transport connection. The exponential map of this connection is

$$\exp_a X = e^{-\frac{1}{2}a^{-1}X} \cdot a = e^{\frac{1}{2}a^{-1}X} a e^{\frac{1}{2}a^{-1}X}.$$

The restriction of the bundle TG^s to P splits as $TG^s|_P = TP \oplus N$ where the "normal bundle" N has over $\rho \in P$ the fiber

$$N_\rho = \{X \in T_\rho G^s : X\rho = \rho X\}.$$

The restriction to N of the exponential map is a diffeomorphism from N onto G^s which preserves the fibers. In Cheeger-Gromoll theory (see **[3]**) this is expressed by saying that P is a soul of G^s.

Returning to the study of the fibration $\pi : G^s \to P$ we give a description of the fibers of π and of the group of all $g \in G$ that preserve the fibers. The tangent map $T\pi : TG^s \to TP$ decreases norms in the sense that $\|(T_a\pi)X\| \leq \|X\|_a$ $(X \in T_aG^s)$. This theorem is based on the inequality $\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|$ valid for bounded linear operators S, T on a Hilbert space with S selfadjoint and invertible **[4]**. The main result of this paper is that given two points in the same fiber G^s_ρ there is a unique geodesic fully contained in G^s_ρ joining them, which is the shortest curve in G^s with the same endpoints. A basic tool of the proof is the above mentioned contraction property of $T\pi$.

In finite dimensional cases, Riemann metrics can be defined on TG^s and we show an example where the canonical connection is the Levi-Civita connection of such a metric. This paper is part of a series devoted to the study of the geometry of several reductive homogeneous spaces which appear naturally in Banach and C^*-algebra theories: the space of idempotents in a C^*-algebra (**[17]**, **[18]**, **[6]**), the space Q_n of n-tuples of idempotents decomposing the identity in a Banach algebra **[5]**, the space of relatively regular elements in a Banach algebra **[8]**. The subset A^+ of G^s of all positive invertible elements of A is also considered in **[7]**, where it is shown that the well-known Segal's inequality (see **[21]**) $\|e^{(X+Y)}\| \leq \|e^{(X/2)}e^Y e^{(X/2)}\|$, where X, Y are selfadjoint elements of A, is equivalent to the property that the exponential map of A^+ increases distances, a property which A^+ shares with Riemannian manifolds with nonpositive curvature. The geometry of some Hilbert homogeneous spaces has been previously studied by P. de la Harpe ([**12**], [**13**]) and Finsler structure of some groups of operators on a Hilbert space has been studied by Atkin ([**1**], [**2**]) who proves some results on uniqueness and minimality of geodesics. The transport equation of Q_n has been independently found by Daleckii and Kato (see [**9**], [**14**] and also [**15**], [**10**]); its geometric meaning, however, was first established in [**5**]. In the case $n = 2$, Q_2 can be identified with the space of all the reflections and its transport equation takes the same form as that of G^s, a phenomenon which will be studied in a forthcoming paper.

1. Preliminaries

Let A be a C*–algebra with 1 represented as an operator algebra in a Hilbert space H. Also denote by $G = G(A)$ the group of invertible elements of A and $G^s = G^s(A)$ the space of invertible selfadjoint elements of G. For each $a \in G^s$ there is a form B^a defined on H by $B^a(x, y) = \langle ax, y \rangle$. The B^a's are hermitian non–degenerate bilinear forms. The B^a–adjoint of $u \in A$ is $u^a = a^{-1}u^*a$. Hence the unitary group U^a of B^a consists of the $u \in G$ with the equivalent properties $u^{-1} = a^{-1}u^*a$ or $(u^*)^{-1}au^{-1} = a$.

In order to study the natural geometry of G^s we introduce the following action of G on G^s:

$$g \cdot a = (g^{-1})^* a g^{-1}.$$

This action fits into the following picture: consider $E = G^s \times H$ as a product bundle over G^s with fiber $E_a = H$ over $a \in G^s$. Then E is a pseudo–Riemannian bundle when each fiber E_a is provided with the form B^a.

E can also be considered as a G–bundle with the action

$$g(a, x) = (g \cdot a, gx).$$

It is clear that this action is isometric on fibers (because $B^{g \cdot a}(gx, gy) = B^a(x, y)$) and that the isotropy group of $a \in G^s$ for the action $g \cdot a$ is the unitary group U^a of the form B^a.

Using $B^{g\cdot a}(gx, gy) = B^a(x, y)$ with $g = \sigma(b)$ the geometric interpretation interpretation of σ is that $\sigma(b)$ an isometry from $E_a = (H, B^a)$ onto $E_b = (H, B^b)$.

In the sequel we denote $G^{s,a}$ the orbit $\{g \cdot a; g \in G\}$ of a.

1.1 PROPOSITION *The orbits $G^{s,a}$ are open and closed in G^s and for each $a \in G^s$, the map*

$$G \to G^{s,a}, \quad g \to g \cdot a$$

is a smooth principal bundle with group U^a.

Proof: It suffices to show that $G \to G^{s,a}$ has a smooth local section near $a \in G^s$. For $b \in G^s$ near a put $\sigma(b) = (b^{-1}a)^{1/2}$. Here $b^{-1}a$ is close to 1 and the square root has the usual meaning (see **[20]** for example). Routine calculations show that

$$\sigma(b) \cdot a = (((b^{-1}a)^{1/2})^{-1})^* a((b^{-1}a)^{1/2})^{-1} = b$$

so that σ is a local section, as needed. This completes the proof of 1.1.

It is readily seen that G^s has a functorial character in the category of C*-algebras and *-homomorphisms. In particular, using Michael's result **[16]** that $G(A) \to G(B)$ is a Serre fibration if $f : A \to B$ is a surjective *-homomorphism, Proposition 1.1 implies that $f : G^s(A) \to G^s(B)$ is onto if and only if every component of $G^s(B)$ contains some element of the image of f. This result is useless in the case when A is the algebra of all bounded linear operators on a Hilbert space H and B is the quotient of A by the ideal of all compact operators (the Calkin algebra of H) since in this case the natural projection $G^s(A) \to G^s(B)$ is onto (**[13]**, p. 197). However in general there is no way of lifting elements and the criterion above may be adequate.

We use $a = \nu\rho$ as the polar decomposition of a with $\nu = |a| = (a^2)^{1/2} > 0$ and with ρ unitary. Since $|a|$ and a commute we have

$$\rho^* = (|a|^{-1}a)^* = a|a|^{-1} = |a|^{-1}a = \rho$$

whence ρ is a selfadjoint unitary element of A, or $\rho^* = \rho^{-1} = \rho$.

2. The canonical connection

Denote by $\mathcal{U}^a$ the Lie algebra of U^a. It is clear that $\mathcal{U}^a$ is a subalgebra of the Lie algebra $\mathcal{G}$ of G and that $\mathcal{G}$ can be identified with A (since G is open in A). In this identification, $\mathcal{U}^a$ corresponds to the set of B^a–antisymmetric elements of A, *i. e.*,

$$\mathcal{U}^a = \{x \in A; a^{-1}x^*a = -x\}.$$

2.1 PROPOSITION *Let S^a denote the set of elements s of A which are B^a–symmetric, i. e., with $a^{-1}s^*a = s$. Then $A = \mathcal{U}^a \oplus S^a$ and the elements of U^a conjugate S^a into itself: if $s \in S^a$ and $g \in U^a$, then $gsg^{-1} \in S^a$.*

Proof: Only the last statement needs a proof:

$$a^{-1}(gsg^{-1})^*a = (a^{-1}(g^{-1})^*a)(a^{-1}s^*a)(a^{-1}g^*a) = gsg^{-1}.$$

2.2 PROPOSITION *For $g \in G$ define $W_g = \{gs; s \in S^a\}$. The the map $g \to W_g \subset T_gG(= A)$ is a distribution of horizontal spaces for a connection on the principal bundle $G \to G^{s,a}$.*

Proof: $(W_g)u = W_{gu}$ for $u \in U^a, g \in G$ is equivalent to $uS^au^{-1} = S^a$, which is shown in Proposition 2.1.

The connection defined by the distribution W_g is the *canonical connection* of the bundle $G \to G^{s,a}$.

2.3 PROPOSITION *If $\gamma(t)$, $u \leq t \leq v$ is a curve in $G^{s,a}$, a curve $\Gamma(t)$ in G is a horizontal lifting of $\gamma(t)$ if and only if $\Gamma(t)$ satisfies the "transport equation"*

$$\dot{\Gamma} = -\frac{1}{2}\gamma^{-1}\dot{\gamma}\Gamma.$$

Proof: Suppose that $\Gamma(t)$ lifts $\gamma(t)$, or $\Gamma(t)\cdot a = \gamma(t)$ or $(\Gamma^{-1}(t))^*a\Gamma^{-1}(t) = \gamma(t)$. Then $\gamma^{-1} = \Gamma a^{-1}\Gamma^*$ and by differentiation we get

$$-\gamma^{-1}\dot{\gamma}\gamma^{-1} = \dot{\Gamma}a^{-1}\Gamma^* + \Gamma a^{-1}\dot{\Gamma}^*$$

or

$$-\gamma^{-1}\dot\gamma = \dot\Gamma a^{-1}\Gamma^*(\Gamma^{-1})^*a\Gamma^{-1} + \Gamma a^{-1}\dot\Gamma^*(\Gamma^{-1})^*a\Gamma^{-1}$$
$$= (\dot\Gamma + M)\Gamma^{-1}$$

where $M = \Gamma a^{-1}(\Gamma^{-1}\dot\Gamma)^*a$. Hence the equation $\dot\Gamma = -(1/2)\gamma^{-1}\dot\gamma\Gamma$ holds if and only if $M = \dot\Gamma$. This in turn is equivalent to

$$\Gamma^{-1}\dot\Gamma = a^{-1}(\Gamma^{-1}\dot\Gamma)^*a,$$

or $\Gamma^{-1}\dot\Gamma \in S^a$ or finally $\dot\Gamma \in W_\Gamma$. This completes the proof.

In the sequel we shall be interested only in solutions Γ of the transport equation with $\Gamma(u) = 1$. These satisfy $\Gamma(t)\cdot\gamma(u) = \gamma(t)$ for all $u \le t \le v$. This Γ will be refered to as the "transport function" of the path $\gamma(t)$ (*cf.* **[5]**, **[10]**, **[14]**, **[15]**, **[18]**). The transport function has the following fundamental property:

2.4 PROPOSITION *If $\gamma(t)$ is a curve in G^s with transport function $\Gamma(t)$ then for $g \in G$ the transport function of $g\cdot\gamma = (g^{-1})^*\gamma g^{-1}$ is $g\Gamma g^{-1}$.*

3. Induced Connections

Suppose $\mathcal{C}$ is a G–manifold ($G = G(A)$) and $\mathcal{C} \to G^s$ is a C^∞ G–Banach bundle, *i.e.*, G operates in a compatible C^∞ way on $\mathcal{C}$ and G^s. A connection D on $\mathcal{C}$ is a *transport connection* if parallel transport in $\mathcal{C}$ along a curve $a(t)$ is given by the transport function of $a(t)$. This means that a section $\sigma(t)$ of $\mathcal{C}$ along $a(t)$, $0 \le t \le 1$, is D-parallel is and only if $\sigma(t) = \Gamma(t)(\sigma(0))$ where $\Gamma(t)$ satisfies $\dot\Gamma = -(1/2)a^{-1}\dot a\Gamma$, $\Gamma(0) = 1$.

3.1 PROPOSITION *Transport connections are G–invariant.*

Proof: Use Proposition 2.4.

We define several transport connections resulting from the systematic use of the transport functions in appropriate contexts.

The bundle E

Let $E = G^s \times H$ as a G–bundle with the action $g(a, x) = (g \cdot a, gx)$ described above in Section 1 and define the connection on E by

$$\frac{Dv}{dt} = \frac{d}{dt}(\Gamma^{-1}(t)v(t))|_{t=0}$$

for any section $v(t) = (a(t), x(t))$ over $a(t)$.

3.2 PROPOSITION *D is a transport connection on E and*

$$D_X v = X(v) + \frac{1}{2}a^{-1}Xv.$$

The curvature of D at $a \in G^s$ is:

$$R(X, Y) = -\frac{1}{4}[a^{-1}X, a^{-1}Y].$$

Next define a Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ on E as follows. For $a \in G^s$ let $a = \nu\rho$ be the polar decomposition of a with $\nu = |a| = (a^2)^{1/2} > 0$ and ρ unitary. We define on the fiber $E_a = H$ the metric

$$\langle\langle x, y\rangle\rangle_a = \langle \nu x, y\rangle = \langle \nu^{1/2}x, \nu^{1/2}y\rangle.$$

Define also a 1–form on G^s with values in A by setting at each $a \in G^s$:

$$S = -\frac{1}{2}a^{-1}[d\rho, \nu]$$

where again $a = \nu\rho$ is the polar decomposition of a.

3.3 PROPOSITION *For any tangent field X on G^s, and any sections x, y of E we have:*

$$X\langle\langle x, y\rangle\rangle - \langle\langle D_X x, y\rangle\rangle - \langle\langle x, D_X y\rangle\rangle = \langle\langle S(X)x, y\rangle\rangle.$$

Proof:

$$
\begin{aligned}
X\langle\!\langle x,y\rangle\!\rangle &- \langle\!\langle \frac{Dx}{dt},y\rangle\!\rangle - \langle\!\langle x,\frac{Dy}{dt}\rangle\!\rangle \\
&= \frac{d}{dt}\langle \nu x, y\rangle - \langle \nu(\dot{x} + \frac{1}{2}a^{-1}\dot{a}x), y\rangle \\
&\quad - \langle \nu x, (\dot{y} + \frac{1}{2}a^{-1}\dot{a}y)\rangle \\
&= \langle \dot{\nu}x, y\rangle + \langle \nu\dot{x}, y\rangle + \langle \nu x, \dot{y}\rangle \\
&\quad - \langle \nu\dot{x}, y\rangle - \frac{1}{2}\langle \nu a^{-1}\dot{a}x, y\rangle \\
&\quad - \langle \nu x, \dot{y}\rangle - \frac{1}{2}\langle \nu x, a^{-1}\dot{a}y\rangle \\
&= \langle \nu(\nu^{-1}\dot{\nu} - \frac{1}{2}a^{-1}\dot{a} - \frac{1}{2}\nu^{-1}\dot{a}a^{-1}\nu)x, y\rangle
\end{aligned}
$$

But

$$
\begin{aligned}
\nu^{-1}\dot{\nu} &- \frac{1}{2}\nu^{-1}\rho(\dot{\rho}\nu + \rho\dot{\nu}) - \frac{1}{2}\nu^{-1}(\dot{\nu}\rho + \nu\dot{\rho})\rho \\
&= \nu^{-1}\dot{\nu} - \frac{1}{2}\nu^{-1}\rho\dot{\rho}\nu - \frac{1}{2}\nu^{-1}\dot{\nu} - \frac{1}{2}\nu^{-1}\dot{\nu} - \frac{1}{2}\dot{\rho}\rho \\
&= -\frac{1}{2}\nu^{-1}\rho\dot{\rho}\nu - \frac{1}{2}\dot{\rho}\rho \\
&= -\frac{1}{2}\nu^{-1}\rho\dot{\rho}\nu + \frac{1}{2}\rho\dot{\rho} \\
&= -\frac{1}{2}a^{-1}(\dot{\rho}\nu - \nu\dot{\rho}) = -\frac{1}{2}a^{-1}[\dot{\rho},\nu],
\end{aligned}
$$

as claimed.

3.4 COROLLARY *Parallel transport on E preserves the metric on curves with $\rho =$ constant.*

The bundle M

We define M as the product bundle $M = G^s \times V$ where V is the space of bounded conjugate bilinear forms on H. The group G acts on V

by $g\beta(x,y) = \beta(g^{-1}x, g^{-1}y)$. If $\beta(t)$ is a curve in M on the curve $a(t)$ we define

$$\frac{D\beta}{dt} = \frac{d}{dt}\Big(\beta(u,v)\Big) - \beta(\frac{Du}{dt}, v) - \beta(u, \frac{Dv}{dt})$$

for any sections u, v in E. The right hand side has the form

$$\begin{aligned}\dot{\beta}(u,v) &+ \beta(\dot{u},v) + \beta(u,\dot{v}) \\ &- \beta(\dot{u},v) - \frac{1}{2}\beta(a^{-1}\dot{a}u, v) \\ &- \beta(u,\dot{v}) - \frac{1}{2}\beta(u, a^{-1}\dot{a}v) \\ =\ & \dot{\beta}(u,v) - \frac{1}{2}\beta(a^{-1}\dot{a}u, v) - \frac{1}{2}\beta(u, a^{-1}\dot{a}v),\end{aligned}$$

which obviously depends only on the values of u, v at each point but not on their derivatives. This means that:

3.5 PROPOSITION *The connection on M is a transport connection with covariant derivative*

$$(D_X\beta)(u,v) = (X(\beta))(u,v) - \frac{1}{2}\beta(a^{-1}Xu, v) - \frac{1}{2}\beta(u, a^{-1}Xv)$$

The bundle $L = G^s \times A$

The elements b in A can be interpreted as bilinear forms by $\beta(u,v) = \langle bu, v\rangle$ and the connection on M induces a connection on $L = G^s \times A$ by

$$\langle \frac{D\sigma}{dt}u, v\rangle = \frac{D\beta}{dt}(u,v)$$

where $\beta(u,v) = \langle \sigma u, v\rangle$.

3.6 PROPOSITION *The connection on L is a transport connection with covariant derivative*

$$D_X\sigma = X(\sigma) - \frac{1}{2}(Xa^{-1}\sigma + \sigma a^{-1}X).$$

The curvature of D satisfies:

$$4R(X,Y)\sigma = \sigma[a^{-1}X, a^{-1}Y] - [Xa^{-1}, Ya^{-1}]\sigma.$$

Proof: The fact that D is a transport connection on L results from calculating for a fixed $b \in A$:

$$\begin{aligned}
\frac{D}{dt}(\Gamma \cdot b) &= \frac{D}{dt}((\Gamma^{-1})^* b\Gamma^{-1}) \\
&= -(\Gamma^{-1})^*\dot{\Gamma}^*(\Gamma^{-1})^* b\Gamma^{-1} - (\Gamma^{-1})^* b\Gamma^{-1}\dot{\Gamma}\Gamma^{-1} \\
&\quad - \frac{1}{2}(\dot{a}a^{-1}(\Gamma^{-1})^* b\Gamma^{-1} + (\Gamma^{-1})^* b\Gamma^{-1}a^{-1}\dot{a}) \\
&= \frac{1}{2}\dot{a}a^{-1}(\Gamma^{-1})^* b\Gamma^{-1} + \frac{1}{2}(\Gamma^{-1})^* b\Gamma^{-1}a^{-1}\dot{a} \\
&\quad - \frac{1}{2}(\dot{a}a^{-1}(\Gamma^{-1})^* b\Gamma^{-1} + (\Gamma^{-1})^* b\Gamma^{-1}a^{-1}\dot{a}) = 0.
\end{aligned}$$

3.7 PROPOSITION *The section $a \to B^a$ in $G^s \times A$ is parallel.*

Proof:

$$\frac{Da}{dt} = \dot{a} - \frac{1}{2}(\dot{a}a^{-1}a + aa^{-1}\dot{a}) = 0.$$

3.8 COROLLARY *The section $a \to (a,a)$ in L is parallel.*

Proof: Since $B^a(x,y) = \langle ax, y\rangle$, B^a corresponds to the tautological section in $G^s \times A$.

The metric $\langle\!\langle\ ,\ \rangle\!\rangle$ in E defines a Finsler structure on the bundle of bilinear forms $M = G^s \times V$, as follows. If $\beta \in M_a$ then

$$\|\beta\|_a = \sup\{|\beta(x,y)|; \langle\!\langle x,x\rangle\!\rangle_a \le 1, \langle\!\langle y,y\rangle\!\rangle_a \le 1\}.$$

With the interpretation of $u \in A$ as the bilinear form $\beta(x,y) = \langle ux, y\rangle$, this translates into a Finsler norm on the bundle $L = G^s \times A$ given explicitly by: for $u \in L_a = A$,

$$\|u\|_a = \|\nu^{-1/2}u\nu^{-1/2}\|$$

($\| \ \|$=ordinary operator norm calculated from $\langle \ , \ \rangle$).

Notice that if $a = \nu\rho = \nu^{-1/2} \cdot \rho \quad (\nu > 0,\ \rho = \text{unitary})$ then the map

$$u \to \nu^{-1/2} \cdot u, \qquad L_\rho \to L_a$$

is an isometry for the norms $\| \ \|_\rho \,(= \| \ \|), \quad \| \ \|_a$. In the sequel length of curves and related concepts refer to this metric through the usual definition

$$\text{Length}(\gamma) = \int \|\dot\gamma(\text{t})\|_{\gamma(\text{t})}\, \text{dt}.$$

The tangent bundle TG^s

The set G^s is open in the real subspace A^s of symmetric elements of A. Hence $TG^s = G^s \times A^s$ is a subbundle of $L = G^s \times A$. Since the covariant derivative in L defined by 3.6 produces symmetric results from symmetric data, we can restrict this connection to TG^s. This is the *canonical connection* on G^s, with covariant derivative defined by

$$D_X Y = X(Y) - \frac{1}{2}(Xa^{-1}Y + Ya^{-1}X)$$

and parallel transport along a curve $a(t)$ in G^s given by the transport function $\Gamma(t)$ of $a(t)$ *acting* on tangent vectors by $\Gamma(t) \cdot X = (\Gamma(t)^{-1})^* X \Gamma(t)^{-1}$. Since the term $Xa^{-1}Y + Ya^{-1}X$ in $D_X Y$ is symmetric in X and Y, the connection in TG^s is a symmetric connection. Similarly, the curvature of TG^s is given by

$$4R(X,Y)Z = Z[a^{-1}X, a^{-1}Y] - [Xa^{-1}, Ya^{-1}]Z.$$

The Finsler structure of $L = G^s \times A$ can be restricted to TG^s. In the sequel we will always consider TG^s as endowed with the resulting structure of Finsler bundle with a transport connection.

Finally we briefly describe the exponential mapping of this connection. Direct computation shows that given $a \in G^s$ and $X \in T_aG^s$, the curve

$\gamma(t) = e^{t\tilde{X}} \cdot a$, where $\tilde{X} = -(1/2)a^{-1}X$, is the geodesic with $\gamma(0) = a$, $\dot{\gamma}(0) = X$. Therefore the exponential mapping is

$$\exp_a X = e^{-a^{-1}X/2} \cdot a.$$

This can also be written as $\exp_a X = a^{1/2} e^{a^{-1/2} X a^{-1/2}} a^{1/2}$.

4. The structure of G^s

Let $P \subset G^s$ be the set of orthogonal reflections of A, i.e., $\rho \in P$ if and only if $\rho^* = \rho = \rho^{-1}$. We define a fibration $\pi : G^s \to P$ by setting $\pi(a) = \rho$ where $a = \nu\rho$ is the polar decomposition of a. As noticed in the preliminaries section, ρ is a selfadjoint unitary, hence an element of P.

Given $\rho \in P$ we write each $u \in A$ as a 2×2 matrix

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

where $u_{11} = pup$, $u_{12} = pu(1-p)$, $u_{21} = (1-p)up$, $u_{22} = (1-p)u(1-p)$, for $p = (\rho+1)/2$ the associated symmetric projection. This decomposes the algebra as $A = A_0 \oplus A_1$ where A_0 consists of the diagonal elements

$$u = \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix}$$

and A_1 consists of the codiagonal elements

$$u = \begin{pmatrix} 0 & u_{12} \\ u_{21} & 0 \end{pmatrix}.$$

Equivalently, $A_0 = \{u; u\rho = \rho u\}$, $A_1 = \{u; u\rho = -\rho u\}$. We say that degree$(u) = 0$ for $u \in A_0$ and degree$(u) = 1$ for $u \in A_1$. Then $A = A_0 \oplus A_1$ is a $\mathbf{Z}_2$–graded algebra.

4.1 PROPOSITION *Denote by G^s_ρ the fibers $\pi^{-1}(\rho)$ of $\pi : G^s \to P$.*

a) $G^s_\rho = \{a \in G^s \cap A_0; a\rho > 0\} = \{\nu\rho; \nu > 0, \nu\rho = \rho\nu\}$.

b) *The group of all $g \in G$ that preserve the fiber G^s_ρ, i.e., $g \cdot a \in G^s_\rho$ for each $a \in G^s_\rho$ is $G \cap A_0$.*

Proof of a): $a \in G^s \cap A_0$ and $a\rho > 0$ imply $a = (a\rho)\rho$ is the polar decompostion of a.

Proof of b): Let $g \in G$ commute with ρ. Then for any $a = \nu\rho \in G^s_\rho$ we have $g \cdot a = (g^{-1})^* \nu\rho g^{-1}$. Then $g \cdot a$ is in A_0 (as a product of degree zero elements) and it is symmetric. Also $(g \cdot a)\rho = (g^{-1})^* \nu g^{-1} > 0$ so that by a) we get $g \cdot a \in G^s_\rho$. Conversely, assume that $g \in G$ acts on G^s_ρ. Then for each $\nu > 0$ with $\nu\rho = \rho\nu$, there exists $\nu' > 0$ with $\nu'\rho = \rho\nu'$ and $g \cdot (\nu\rho) = \nu'\rho$. Decomposing $g^{-1} = h_0 + h_1$ with $h_0 \in A_0$ and $h_1 \in A_1$ we get

$$\begin{aligned} \nu'\rho = g \cdot (\nu\rho) &= (h_0^* + h_1^*)\nu\rho(h_0 + h_1) \\ &= (h_0^* + h_1^*)\nu(h_0 - h_1)\rho, \end{aligned}$$

so that after cancelling ρ and comparing terms of the same degree we get

$$h_0^*\nu h_0 - h_1^*\nu h_1 = \nu' \qquad h_0^*\nu h_1 - h_1^*\nu h_0 = 0.$$

Taking $\nu = 1$ it follows that $h_0^* h_0 = \nu' + h_1^* h_1 > 0$ and h_0 is invertible. But the equality $h_0^*\nu h_1 = h_1^*\nu h_0$ can not hold for all $\nu > 0$ commuting with ρ unless $h_1 = 0$. In fact consider the example

$$\nu = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

and write

$$h_0 = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix} \quad h_1 = \begin{pmatrix} 0 & h_{12} \\ h_{21} & 0 \end{pmatrix}.$$

Then from $h_0^*\nu h_1 = h_1^*\nu h_0$ we get

$$h_{11}^*\alpha h_{12} = h_{21}^*\beta h_{22}$$

and since we can take $\alpha, \beta > 0$ arbitrary real numbers, we get $h_{11}^* h_{12} = 0$ and $h_{21}^* h_{22} = 0$. Cancelling h_{11}^* and h_{22} we conclude that $h_{12} = 0$, $h_{21} = 0$ and therefore $h = 0$. This means that g^{-1} (whence g) has degree 0 and the proof is complete.

The restriction to P of the bundle TG^s splits as a sum $TG^s|_P = TP \oplus N$ where the "normal" bundle N is defined by $N_\rho = \{x \in T_\rho G^s; x\rho = \rho x\}$.

4.2 THEOREM *Let $\Xi : N \to G^s$ be the restriction to N of the exponential mapping of G^s, so that $\Xi(\rho, X) = e^{-\rho X/2} \cdot \rho$. Then Ξ is a diffeomorphism satisfying $\Xi(N_\rho) = G^s_\rho$.*

Proof: The inverse of Ξ is given at $a = \nu\rho$ by $\Xi^{-1}(a) = (\rho, \rho \ln \nu)$.

We close this section with the remark that geodesics in a fiber with given endpoints are unique. This follows from the fact that positive elements have unique symmetric logarithms. In fact, if $x \in G^s_\rho$ and $H = H_+ \oplus H_-$ with $H_\pm = \{x; \rho x = \pm x\}$, then

$$a = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}$$

can be written in a unique way as $a = e^{\tilde{X}} \cdot \rho$ where

$$\tilde{X} = \begin{pmatrix} -\frac{1}{2}X_+ & 0 \\ 0 & \frac{1}{2}X_- \end{pmatrix},$$

and $X_\pm$ symmetric. So there is a unique geodesic joining ρ with a. For arbitrary $b, a \in G^s_\rho$, operate first with a convenient $g \in G \cap A_0$ to reduce to the case $b = \rho$.

5. Projecting on the base

The basic fact of this section is the following.

5.1 THEOREM *The tangent map $T\pi : TG^s \to TP$ decreases norms.*

Proof: We want to prove that

$$\|T_a \, \pi X\| \leq \|X\|_a$$

for all $a \in G^s$. Let $a(t)$ be a curve in G^s and $X = \dot{a}(t)$. Let $\rho(t) = \pi(a(t))$ and let $\Gamma(t)$ be the transport function of $\rho(t)$. Finally define $a_1(t) = \Gamma(t) \cdot a(0)$. Since $\pi(a(t)) = \pi(a_1(t))$ ($\Gamma(t)$ is unitary) we get that $X_2 = \dot{a}(0) - \dot{a}_1(0)$ is tangent to the fiber $\pi^{-1}(\rho(0))$. Next calculate at $t = 0$:

$$X_1 = \dot{a}_1 = \frac{d}{dt}(\Gamma(t) \cdot a(0)) = \frac{1}{2}(-\rho\dot{\rho}a + a\rho\dot{\rho}).$$

Writing at $t = 0$ the polar decomposition $a = \nu\rho = \rho\nu$ we get

$$X_1 = \frac{1}{2}(-\rho\dot{\rho}\rho\nu + \nu\rho\rho\dot{\rho}) = \frac{1}{2}(\dot{\rho}\nu + \nu\dot{\rho}).$$

Then calculate

$$\begin{aligned}
\|X\|_a &= \|\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}\| \\
&= \|\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}}\| \\
&= \|\frac{1}{2}(\nu^{-\frac{1}{2}} \dot{\rho} \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \dot{\rho} \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}}\|.
\end{aligned}$$

Recall the inequality ([4]):

$$\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|$$

valid for any symmetric invertible operator S and any operator T. This reduces the proof of the theorem to the inequality

$$\|\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}\| \geq \|\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}}\|.$$

But

$$\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}} = \nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}}$$

is the decompostion of $\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}$ in degree 1 and degree 0 components determined by $\rho(0)$. This is clear because $\rho\dot{\rho} = -\dot{\rho}\rho$ and X_2 is tangent to $G^S_{\rho(0)}$. Therefore if we write

$$\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}} = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$$

$$\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}} = \begin{pmatrix} 0 & \beta^* \\ \beta & 0 \end{pmatrix}$$

$$\nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$$

then clearly

$$\|\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}\| \geq \|\beta\| = \|\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}}\|.$$

5.2 THEOREM *A geodesic of length less than π contained in P is the shortest curve in G^s joining its endpoints.*

Proof: Let γ be the geodesic in P joining ρ_0 and ρ_1 and let δ be any other curve joining ρ_0 and ρ_1. Then $\delta_1 = \pi(\delta)$ is contained in P and according to Theorem 5.1, the length of δ_1 does not exceed the length of δ. Then observing that the Finsler metric of G^s restricted to P is given by ordinary operator norm, a direct application of **[18]** gives the desired minimality and uniqueness.

6. Geodesics in a fiber

Suppose $a(t)$, $0 \leq t \leq 1$ is a curve in G^s with $\pi(a(0)) = a(1)$.

Denote $\rho(t) = \pi(a(t))$, $\nu(t) = a(t)\rho(t)$, and $\Gamma(t)$ the transport function of $\rho(t)$. Next define $\sigma(t) = \Gamma^{-1}(t)a(t)\Gamma(t)$. Since $\Gamma(t)$ is unitary, the polar decomposition of σ is

$$\sigma = (\Gamma^{-1}\nu\Gamma)(\Gamma^{-1}\rho\Gamma)\ ,$$

or $\pi(\sigma) = \Gamma^{-1}\rho\Gamma = \rho(0)$ for each t. This means that σ is a curve in $G^s_{\rho(0)}$.

Observe that σ has the same endpoints as a because

$$\sigma(0) = \Gamma^{-1}(0)a(0)\Gamma(0) = a(0)$$

and by the hypothesis $\pi(a(0)) = a(1)$ we have $\rho(1) = a(1)$ and therefore $\sigma(1) = \Gamma^{-1}(1)a(1)\Gamma(1) = \Gamma^{-1}(1)\rho(1)\Gamma(1) = \rho(0) = \rho(1) = a(1)$.

We claim that

$$(\P) \qquad \|\dot{\sigma}\|_\sigma \leq \|\dot{a}\|_a\ .$$

First (use $\rho\dot\rho = -\dot\rho\rho,\ a = \nu\rho$, etc.):

$$\begin{aligned}\dot\sigma &= -\Gamma^{-1}(-\frac{1}{2}\rho\dot\rho)a\Gamma + \Gamma^{-1}a(-\frac{1}{2}\rho\dot\rho)\Gamma + \Gamma^{-1}\dot a\Gamma \\ &= \Gamma^{-1}(\frac{1}{2}(\rho\dot\rho a - a\rho\dot\rho) + \dot a)\Gamma \\ &= \Gamma^{-1}\frac{\rho\dot\nu + \dot\nu\rho}{2}\Gamma\end{aligned}$$

and therefore

$$\begin{aligned}\|\dot\sigma\|_\sigma &= \|(\Gamma^{-1}\nu^{-1/2}\Gamma)\dot\sigma(\Gamma^{-1}\nu^{-1/2}\Gamma)\| \\ &= \|\Gamma^{-1}\nu^{-1/2}\frac{\rho\dot\nu + \dot\nu\rho}{2}\nu^{-1/2}\Gamma\| \\ &= \frac{1}{2}\|\nu^{-1/2}(\rho\dot\nu + \dot\nu\rho)\nu^{-1/2}\|\ .\end{aligned}$$

On the other hand, $a = \nu\rho = \rho\nu$ gives

$$\dot a = \frac{1}{2}(\rho\dot\nu + \dot\nu\rho) + \frac{1}{2}(\dot\rho\nu + \nu\dot\rho)$$

and then

$$\|\dot a\|_a = \frac{1}{2}\|\nu^{-1/2}(\rho\dot\nu + \dot\nu\rho)\nu^{-1/2} + \nu^{-1/2}(\dot\rho\nu + \nu\dot\rho)\nu^{-1/2}\|$$

But in the matrix decomposition at each $\rho(t)$

$$\begin{aligned}\nu^{-1/2}(\rho\dot\nu + \dot\nu\rho)\nu^{-1/2} &= \begin{pmatrix}\alpha & 0\\ 0 & \gamma\end{pmatrix} \\ \nu^{-1/2}(\dot\rho\nu + \nu\dot\rho)\nu^{-1/2} &= \begin{pmatrix}0 & \beta^*\\ \beta & 0\end{pmatrix}\end{aligned}$$

(because the former commutes with ρ and the latter anticommutes with ρ). Hence

$$\left\|\begin{pmatrix}\alpha & \beta^*\\ \beta & \gamma\end{pmatrix}\right\| \geq \left\|\begin{pmatrix}\alpha & 0\\ 0 & \gamma\end{pmatrix}\right\|$$

implies $\|\dot a\|_a \geq \|\dot\sigma\|_\sigma$. This is inequality (¶) and the claim is proved.

This inequality shows that:

6.1 PROPOSITION. *For any curve joining $a \in G^s$ with $\pi(a)$, there is a shorter curve in the fiber $G^s_{\pi(a)}$ with the same endpoints.*

The following technical result is needed in the proof of Theorem 6.3:

6.2 LEMMA. *Let p be a rank 1 orthogonal projection in the Hilbert space H, $a : H \to H$ positive definite, $X : H \to H$ selfadjoint. Then*

$$\|pa^{1/2}Xa^{1/2}p\| \leq \|pap\| \, \|X\| \, .$$

Proof: Decompose $H = \mathbf{C}e \oplus H_1$ where $\|e\| = 1$, $p(e) = e$, and $H_1 = \ker(p)$. Then we have matrix representations

$$a^{1/2} = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$$
$$X = \begin{pmatrix} \xi & \eta^* \\ \eta & \theta \end{pmatrix}$$

where A, ξ are scalars, $B \in H_1$ and $B^* : H_1 \to \mathbf{C}$ is the functional $B^*(h) = \langle h, B \rangle$, and θ, C are operators in H_1. Define also a bilinear map $F : H \times H \to \mathbf{C}$ by $F(u,v) = \langle Xu, v \rangle$. Then calculating we find that the (1,1) entry W_{11} of $W = a^{1/2}Xa^{1/2}$ is $F(Ae + B, Ae + B)$. Then

$$\|W_{11}\| \leq \|F\| \, \|Ae + B\|^2 = \|X\| \, \|Ae + B\|^2 = \|X\|(A^2 + |B|^2) \, .$$

But

$$a = (a^{1/2})^2 = \begin{pmatrix} A^2 + B^*B & AB^* + B^*C \\ BA + CB & BB^* + C^2 \end{pmatrix}$$

and so

$$\|W_{11}\| \leq \|X\| \, \|a_{11}\| \, ,$$

as claimed.

6.3 THEOREM. *The unique geodesic in G_ρ^s joining two points $a, b \in G_\rho^s$ is the shortest curve in G^s joining a and b.*

Proof: We consider first the case where $b = \rho$. Let $\omega(t)$, $0 \le t \le 1$ be a curve joining ρ and a, and $\gamma(t) = e^{t\widetilde{X}} \cdot \rho$, $0 \le t \le 1$, the geodesic in G_ρ^s joining the same endpoints where $X = \dot\gamma(0) \in T_\rho G_\rho^s$ and $\widetilde{X} = -\frac{1}{2}\rho X$. We will show that

$$\text{Length}(\omega) \ge \text{Length}(\gamma) .$$

By 6.1 we may assume that ω is fully contained in G_ρ^s. We handle first the case $\rho = 1$.

By changing the representation if necessary, we can find $e \in H$ with $Xe = \lambda e$, $\|e\| = 1$ and $|\lambda| = \|X\|$. Next, we decompose H as $H = \mathbf{C}e \oplus \mathbf{C}e^\perp$ and therefore we can obtain by compression to $\mathbf{C}e$ two curves γ_{11} and ω_{11} defined as the (1,1) entries of the matrices of γ and ω in the decomposition $H = \mathbf{C}e \oplus \mathbf{C}e$. By 6.2 we have $\text{Length}(\omega_{11}) \le \text{Length}(\omega)$. Also, $\gamma_{11}(t) = (e^{t\widetilde{X}} \cdot \rho) = e^{t\lambda}$ and

$$\|\dot\gamma_{11}\|_{\gamma_{11}} = |e^{t\lambda}\lambda|_{\gamma_{11}} = |e^{-t\lambda/2}e^{t\lambda}e^{-t\lambda/2}\lambda| = |\lambda|$$

so that

$$\text{Length}(\gamma_{11}) = |\lambda| = \|X\| = \text{Length}(\gamma) .$$

Since $\omega_{11}(t) > 0$ we can calculate

$$\begin{aligned}\text{Length}(\omega_{11}) &= \int_0^1 \|\dot\omega_{11}(t)\|_{\omega_{11}(t)} dt \\ &\int_0^1 |\omega_{11}^{-1/2}(t)\dot\omega_{11}(t)\omega_{11}^{-1/2}(t)| dt \\ &= \int_0^1 |\dot\omega_{11}(t)/\omega_{11}(t)| dt \ge |\log \omega_{11}(t)|_0^1| = |\lambda|\end{aligned}$$

since $\omega_{11}(1) = \gamma_{11}(1) = e^\lambda$, $\omega_{11}(0) = \gamma_{11}(0) = 1$. This shows that γ is minimal in the case $\rho = 1$.

Consider next an arbitrary ρ and decompose $H = H_+ \oplus H_-$ where $H_\pm = \{x\,;\ \rho x = \pm x\}$. Then

$$X = \begin{pmatrix} X_+ & 0 \\ 0 & X_- \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} -\frac{1}{2}X_+ & 0 \\ 0 & +\frac{1}{2}X_- \end{pmatrix}$$

and

$$\gamma(t) = e^{t\widetilde{X}} \cdot \rho = \begin{pmatrix} e^{tX_+} & 0 \\ 0 & -e^{-tX_-} \end{pmatrix}.$$

Similarly,

$$\omega(t) = \begin{pmatrix} \omega_+(t) & 0 \\ 0 & \omega_-(t) \end{pmatrix}.$$

But,

$$\|X\| = \|X_+\| \quad \text{or} \quad \|X\| = \|X_-\|$$

and

$$\|\dot{\omega}(t)\|_{\omega(t)} \geq \|\dot{\omega}_\pm(t)\|_{\omega_\pm(t)}$$

so that by choosing the half where X keeps its norm we are (up to sign) in the case $\rho = 1$, and the proof is complete.

To complete the proof, operate with an element of $G \cap A_0$ to reduce the general case to $b = \rho$.

7. An example

We consider now the algebra A of linear endomorphisms of the Hilbert space $\mathbf{C}^2$ with the standard inner product. Then $G = GL(2,\mathbf{C})$ and G^s has three connected components determined by signature. Denote G_1^s the component consisting of the positive definite elements of A. The level manifolds $M_h = \{a; \det(a) = h\}$ of the determinant function $\det : G_1^s \to \mathbf{R}^+$ form a smooth foliation with three dimensional leaves. Also the rays $N_a = \{ra\,;\ r > 0\}$ with $a \in M_1$, form a one dimensional foliation and $\{M_h\}$ is transversal to $\{N_a\}$. The leaves M_h are the orbits of the action $g\,.\,a = (g^{-1})^* a g^{-1}$ of the subgroup $SL(2,\mathbf{C}) \subset GL(2,\mathbf{C})$ and the leaves N_a are the orbits of the center $\{z1\,;\ z \neq 0\}$ of $GL(2,\mathbf{C})$.

Since a curve through $a(0) = 1$ with $\det(a(t)) = 1$ satisfies $\operatorname{tr}(\dot{a}(0)) = 0$, by translation we have $\operatorname{tr}(a^{-1}\dot{a}) = 0$ for all curves in M_h. Then the solution Γ of the transport equation $\dot{\Gamma} = -\frac{1}{2}a^{-1}\dot{a}\Gamma$ is contained in $SL(2,\mathbf{C})$. Therefore the canonical connection on TG_1^s preserves the leaves M_h (in the sense that D_XY is tangent to M_h whenever both X and Y are), and these leaves are totally geodesic.

Introduce a Riemannian metric on G_1^s by $(X,Y)_a = \mathrm{tr}(a^{-1}Xa^{-1}Y)$ for $X, Y \in T_aG_1^s$. Writing

$$(X,Y)_a = \mathrm{tr}((a^{-1/2}Xa^{-1/2})(a^{-1/2}Ya^{-1/2}))$$

shows immediately that $(X,Y)_a$ is positive definite. The foliations $\{M_h\}$ and $\{N_a\}$ are orthogonal for $(\ ,\ )$.

7.1 PROPOSITION. *The canonical connection in TG_1^s is the Levi-Civita connection of the Riemann metric $tr(a^{-1}Xa^{-1}Y)$ and $GL(2,\mathbf{C})$ acts isometrically on G_1^s.*

Proof: We already observed that the canonical connection is symmetric. Using 3.6 one verifies that, for X, Y, Z tangent fields, it holds that

$$Z(X,Y) = (D_ZX,Y) + (X,D_ZY)$$

and this completes the proof.

The tangent space T_1M_1 to det $= 1$ at $a = 1$ is the space of symmetric 2×2 matrices with trace zero. Using

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

we can write the arbitrary element

$$X = \begin{pmatrix} y & z+ix \\ z-ix & -y \end{pmatrix}$$

in T_1M_1 as

$$(\dagger) \qquad X = -i(xI + yJ + zK)$$

(x, y, z are *real*). Further, each $g \in SU(2)$ has the form

$$g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} , \quad |\alpha|^2 + |\beta|^2 = 1$$

and writing $\alpha = s + ui, \ \beta = v + wi$ we can expand g as

$$g = s + uI + vJ + wK \ .$$

The condition $|\alpha|^2 + |\beta|^2 = s^2 + u^2 + v^2 + w^2 = 1$ implies

$$g^{-1} = s - uI - vJ - wK = g^*$$

and therefore

$$g \cdot X = gXg^{-1} \ .$$

This shows that the action of $SU(2)$ on T_1M_1 corresponds to the action by inner automorphism of quaternions g with $|g| = 1$ on the 3-space of purely imaginary quaternions. Then with elements of $SU(2)$ we can obtain any rotation of $\mathbf{R}^3$ identified to T_1M_1 through $X \to (x, y, z)$ as in (†). In particular any plane in T_1M_1 can be mapped onto any other plane.

Observe next that $SU(2)$ operates isometrically on M_1 and leaves 1 fixed. Hence the action of $SU(2)$ leaves sectional curvature $K(X, Y) = (R(X, Y)Y, X)$ invariant. This shows that the sectional curvature in TM_1 is the same for all planes in TM_1. Then operating with $g \in SL(2, \mathbf{C})$ we conclude the M_1 has constant sectional curvature. For any pairs $X, Y \in T_1M_1$, we can calculate

$$4(R(X, Y)Y, X) = \operatorname{tr}((XY)^2) - \operatorname{tr}(X^2Y^2)$$

so that taking $X = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix}$ we can verify that $(X, X) = (Y, Y) = 1, \ (X, Y) = 0$ and therefore the sectional curvature of M_1 is

$$\frac{1}{4}(\operatorname{tr}(XY)^2 - \operatorname{tr}(X^2Y^2)) = -\frac{1}{4} \ .$$

More generally (with the same proof!):

7.2 PROPOSITION. *The submanifolds $M_h \subset G_1^s$ defined for each $h > 0$ by* det $= h$ *have constant sectional curvature* $-1/4\sqrt{h}$.

8. Appendix

There is an alternative way of obtaining the transport function of γ in terms of multiplicative integrals (see **[19]**, **[11]**, **[22]**). Consider a curve $\gamma(t)$, $u \leq t \leq v$ in G^s. Assuming $\gamma(t)$ continuous we can find a partition $\Pi = \{u = t_0 \leq t_1 \leq \cdots \leq t_n = v\}$ with $\gamma(t_i)$ and $\gamma(t_{i+1})$ close for all i. Next define

$$P_\Pi = \left(\gamma(t_n)^{-1}\gamma(t_{n-1})\right)^{1/2} \cdots \left(\gamma(t_2)^{-1}\gamma(t_1)\right)^{1/2}\left(\gamma(t_1)^{-1}\gamma(t_0)\right)^{1/2}$$

which makes sense because $\gamma(t_{i+1})^{-1}\gamma(t_i)$ is close to 1 for all i. Since

$$\left(\gamma(t_{i+1})^{-1}\gamma(t_i)\right)^{1/2} \cdot \gamma(t_i) = \gamma(t_{i+1})$$

(proof of Proposition 1.1) we get $P_\Pi \cdot \gamma(u) = \gamma(v)$. Taking limits on the partition (assume that the curve is smooth) we can define the multiplicative integral

$$P(v,u) = \lim_{\Pi} P_\Pi$$

and then

$$P(v,u) \cdot \gamma(u) = \gamma(v).$$

From the definition of P we see also that for $u \leq w \leq v$:

$$P(w,v)P(v,u) = P(w,u)$$

or

$$P(w,v) = P(w,u)P(v,u)^{-1} = P(w)P(v)^{-1}$$

where we abbreviate $P(t) = P(t,u)$ with u the left endpoint.

8.1 PROPOSTION *Given a smooth curve $\gamma(t)$, $u \le t \le v$ in G^s, the horizontal lifting $\Gamma(t)$ of $\gamma(t)$ with initial condition $\Gamma(u) = 1$ is given by $\Gamma(t) = P(t,u)$.*

Proof: We will see that $P(t,u)$ satisfies the transport equation $\dot\Gamma = -(1/2)\gamma^{-1}\dot\gamma\Gamma$. For that approximate the curve $\gamma(t)$ by a piecewise linear curve $\tau(t)$ joining $\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_n)$ so that between t_i and t_{i+1} we have $\tau(t) = \gamma(t_i) + s(\gamma(t_{i+1} - \gamma(t_i)$ where $s = (t-t_i)/(t_{i+1} - t_i)$. Abbreviate $a = \gamma(t_i)$, $b = \gamma(t_{i+1})$. Then

$$\tau = a + s(b-a) = a(1 + sa^{-1}(b-a))$$
$$\dot\tau = \dot s(b-a)$$

so that letting $c = a^{-1}(b-a)$ we can write

$$\tau = a(1+sc)$$
$$\tau^{-1}(b-a) = (1+sc)^{-1}c$$

and

$$\tau^{-1}\dot\tau = \dot s(1+sc)^{-1}c.$$

Then the function $T_i(t) = (1+sc)^{-1/2}$ satisfies $T_i^2(t) = (1+sc)^{-1}$ and

$$\dot T_i T_i + T_i \dot T_i = -(1+sc)^{-1}\dot s c(1+sc)^{-1}$$

so

$$\dot T_i T^{-1} + T_i \dot T_i T_i^{-2} = -(1+sc)^{-1}\dot s c = -\tau^{-1}\dot\tau.$$

Therefore

$$\dot T_i T_i^{-1} = -\frac{1}{2}\tau^{-1}\dot\tau - \frac{1}{2}[T_i, \dot T_i]T_i^{-2}.$$

Now at $t = t_i$ we have $T_i = 1$ and then $[T_i, \dot T_i]T_i^{-2} = 0$ there. Hence if a and b are close then:

$$\dot T_i T_i^{-1} = -\frac{1}{2}\tau^{-1}\dot\tau - K$$

with K small. Define now for $t_i \leq t \leq t_{i+1}$ the function

$$T_\Pi(t) = T_i(t)T_{i-1}(t_i)T_{i-2}(t_{i-1})\dots T_0(t_1).$$

Taking limits on the partition Π we get the function

$$T_1 = \lim_\Pi T_\Pi$$

and the identities

$$\gamma = \lim_\Pi \tau, \qquad 0 = \lim_\Pi K.$$

Hence T_1 satisfies

$$\dot{T}_1 T_1^{-1} = -\frac{1}{2}\gamma^{-1}\dot{\gamma}.$$

But $T_1 = P$. In fact, let us calculate:

$$\begin{aligned} T_i(t_{i+1}) &= (1+c)^{-1/2} \\ &= (1 + a^{-1}(b-a))^{-1/2} \\ &= (1 + a^{-1}b - 1)^{-1/2} \\ &= (a^{-1}b)^{-1/2} = (b^{-1}a)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} T_\Pi(t_n) &= T_{n-1}(t_n)T_{n-2}(t_{n-1})\dots \\ &= \Big(\gamma(t_n)^{-1}\gamma(t_{n-1})\Big)^{-1/2}\Big(\gamma(t_{n-1})^{-1}\gamma(t_{n-2})\Big)^{-1/2}\dots \end{aligned}$$

and therefore $T_1 = \lim T_\Pi = P$ as claimed.

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