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Conditional Expectations and Operator Decompositions

HORACIO PORTA AND LÁZARO RECHT

Abstract: For a C^*-algebra A with a conditional expectation $\Phi : A \to A$ onto a subalgebra B we have the linear decomposition $A = B \oplus H$ where $H = \ker(\Phi)$. Since Φ preserves adjoints, it is also clear that a similar decomposition holds for the selfadjoint parts: $A^s = B^s \oplus H^s$ (we use $V^s = \{a \in V; a^* = a\}$ for any subspace V of A). Apply now the exponential function to each of the three terms A^s, B^s, and H^s. The results are: the set G^+ of positive invertible elements of A, the set B^+ of positive invertible elements of B, and the set $C = \{e^h; h^* = h, \Phi(h) = 0\}$, respectively.

We consider here the question of lifting the decomposition $A^s = B^s \oplus H^s$ to the exponential sets. Concretely, is every element of G^+ the product of elements of B^+ and C, respectively, just as any selfadjoint element of A is the sum of selfadjoint elements of B and H? The answer is yes in the following sense: Each $a \in G^+$ is the positive part of a product bc of elements $b \in B^+$ and $c \in C$, and both b and c are uniquely determined and depend analytically on a. This can be rephrased as follows: The map $(b, c) \to (bc)^+$ is an analytic diffeomorphism from $B^+ \times C$ onto G^+, where for any invertible $x \in A$ we denote with x^+ the positive square root of xx^*. This result can be expressed equivalently as: The map $(b, c) \to bcb$ is a diffeomorphism between the same spaces.

Notice that combining the polar decomposition with these results we can write every invertible $g \in A$ as $g = bcu$, where $b \in B^+$, $c \in C$, and u is unitary. This decomposition is unique and the factors b, c, u depend analytically of g. In the case of matrix algebras with $\Phi =$ trace/dimension, the factorization corresponds to $g = |\det(g)|cu$ with $c > 0, \det(c) = 1$, and u unitary.

This paper extends some results proved by G. Corach and the authors in [2]. Also, Theorem 2 states that the reductive homogeneous space resulting from a conditional expectation satisfies the regularity hypothesis introduced by L. Mata-Lorenzo and L. Recht in [5], Definition 11.1. The situation considered here is the "general context" for regularity indicated in the introduction of the last mentioned paper.

Key words: *Conditional expectation, reductive homogeneous space, geodesic, exponential map, Finsler metric*

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Let A be a C^*-algebra with 1. Denote by G the group of invertible elements of A, and by G^+ the set of positive elements of G. Recall ([2]) that G^+ has a canonical structure of reductive homogeneous space with group action $L_g a = (g^{-1})^* a g^{-1}$ for $g \in G$ and $a \in G^+$. The resulting G-invariant connection has covariant derivative

$$\frac{DY}{dt} = \frac{dY}{dt} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}Y + Y\gamma^{-1}\dot{\gamma})$$

where $Y(t)$ is a tangent field along the curve $\gamma(t)$ in G^+. The corresponding exponential has the formula

$$\exp_a X = e^{\frac{1}{2}Xa^{-1}} a e^{\frac{1}{2}a^{-1}X}$$

for $X \in T(G^+)_a$.

The space G^+ carries also the G-invariant Finsler metric $\|X\|_a = \|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\|$ where $\|\ \|$ is the original norm of A (the inclusion $G^+ \subset A$ permits to identify the tangent space $T(G^+)_a$ with the real space of selfadjoint elements X of A). The Finsler distance in the metric space G^+ is given by

$$\operatorname{dist}(a,b) = \|\ln(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|$$

for $a, b \in G^+$.

Throughout we will assume that A is provided with a fixed conditional expectation $\Phi : A \to A$ (see [4] for the definition and properties of conditional expectations in a C^*-algebra). We denote by $B = \Phi(A)$ and $H = \text{kernel}(\Phi)$ the image and the kernel of the conditional expectation. B is a C^*-subalgebra of A. Also B^+ denotes $B^+ = B \cap G^+$ and for any $V \subset A$ we use $V^s = \{a \in V; a^* = a\}$. Finally we denote $C = \{e^h; h \in H^s\}$.

Relative to Φ we can define a normal bundle N for the immersion $B^+ \subset G^+$ as follows: N is the set of pairs $(b, X) \in B^+ \times A$ such that $b^{-\frac{1}{2}} X b^{-\frac{1}{2}} \in H^s$, with norm $\|(b,X)\| = \|X\|_b = \|b^{-\frac{1}{2}} X b^{-\frac{1}{2}}\|$. The fiber of N over $b \in B^+$ will be denoted by N_b and its elements (b,X) will be refered to as its "vector" components X.

We will say that a geodesic $\gamma(t)$ in G is *normal at* $a \in B^+$ if $\gamma(0) = a$ and $\dot\gamma(0) \in N_a$.

1. Theorem. *If $\alpha(t)$, $\beta(t)$ are normal geodesics at a, $b \in B^+$ respectively, then*

$$t \to \operatorname{dist}(\alpha(t), \beta(t))$$

is a non-decreasing convex function of $t \geq 0$.

Proof. Since the group action L_g by g invertible in B preserves the whole structure in A, G^+, and N we may assume without loss of generality that $a = 1$. Then we can write $\alpha(t) = e^{tX}$, $\beta(t) = b^{\frac{1}{2}} e^{tY} b^{\frac{1}{2}}$ for X, $Y \in H^s$. Abbreviate

$$\begin{aligned} \gamma(t) &= e^{-\frac{1}{2}tX} b^{\frac{1}{2}} e^{tY} b^{\frac{1}{2}} e^{-\frac{1}{2}tX} \\ f(t) &= \operatorname{dist}(\alpha(t), \beta(t)) = \|\ln \gamma(t)\|. \end{aligned}$$

The function $f(t)$ is a convex function of $t \in \mathbf{R}$ by Theorem 2 in [3]. We will show that the right derivative $f'(0^+)$ is non-negative, and this suffices to conclude that $f(t)$ is non-decreasing for $t \geq 0$. For that choose a real bounded linear function ϕ on A^s with the properties:

(i) $\|\phi\| = 1$

(ii) $\phi(\ln b) = \|\ln b\|$.

Replacing ϕ by $\phi \circ \Phi$ we may assume that ϕ satisfies also:

(iii) $\phi(Z) = 0$ for all $Z \in H^s$.

We will compute $\frac{d}{dt}\phi(\ln \gamma(t))|_{t=0}$ by approximating the ln function by polynomials.

Assume then that $\{p_n(x)\}_n$ is a sequence of polynomials with $p_n(x) \to \ln x$ and $p_n'(t) \to 1/x$ uniformily on an intervall $[x_0, x_1]$ with $x_0 > 0$ and such that $[x_0, x_1]$ contains the spectrum of each $\ln(\gamma)$ for $0 \le t \le 1$. Write explicitely $p_n(x) = \sum c_{nk} x^k$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} p_n\big(\gamma(t)\big)\Big|_{t=0} = \sum c_{nk} \frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t)^k\big)\Big|_{t=0}$$

and each derivative $(\mathrm{d}(\gamma(t^k)/\mathrm{d}t)|_{t=0}$ is a linear combination of terms of the form

$$(b^{\frac{1}{2}})^j X (b^{\frac{1}{2}})^l \quad \text{and} \quad (b^{\frac{1}{2}})^j Y (b^{\frac{1}{2}})^l.$$

But these terms are in H because $\Phi(b_1 h b_2) = b_1 \Phi(h) b_2 = 0$ for $h \in H$ and $b_1, b_2 \in B$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi\big(\ln \gamma(t)\big)\Big|_{t=0} = \phi\Big(\frac{\mathrm{d}}{\mathrm{d}t} \ln \gamma(t)\Big)\Big|_{t=0} = 0.$$

Notice that $\phi(\ln \gamma(t)) \le f(t)$ because $\|\phi\| = 1$ and that $\phi(\ln \gamma(0)) = \phi(\ln b) = \|\ln b\| = f(0)$. Thus by convexity of $f(t)$ we have $f'(0^+) \ge \frac{\mathrm{d}}{\mathrm{d}t}\phi(\ln \gamma(t))|_{t=0} = 0$ and therefore $f(t)$ is non-decreasing for $t \ge 0$. □

The following is the main result of this paper:

2. Theorem. *The map $(b, X) \to \exp_b(X)$ is an analytic diffeomorphism of N onto G^+.*

The proof requires some preliminaries. Denote by $\Theta : N \to G^+$ the map $\Theta(X) = \Theta(b, X) = \exp_b(X)$. Notice that Θ is injective by Theorem 1. Theorem 2 says that Θ is also onto and a diffeomorphism. Actually it is analytic, as the proof shows.

3. Proposition. *For each $\varepsilon > 0$ small enough, this image Ω^ε under Θ of the disk bundle $N^\varepsilon = \{(b, X) \in N; \|X\|_b < \varepsilon\}$ is an open neighborhood of B^+ in G^+. The open set Ω^ε is invariant under the action by invertible elements of B and contains the metric tube $W^\delta = \{a \in A^+;\ \mathrm{dist}(a, B^+) \le \delta\}$ for appropriate $\delta > 0$.*

Proof. By definition and with the natural identifications, the tangent map to Θ at $b = 1$, $X = 0$ is the identity map and therefore by equivariance of Θ the proposition follows. □

Next we define $\eta_\lambda : \Theta(N) \to \Theta(N)$ as the map that corresponds to multiplication by λ in N, that is

$$\eta_\lambda(\Theta(b, X)) = \Theta(b, \lambda X).$$

Here are some basic properties of the maps η_λ:

4.i. η_1 is the identity and η_0 is the retraction $\exp_b(X) \to b$ from $\Theta(N)$ onto B^+ (the image under Θ of the zero section of N).

4.ii. The η_λ with $\lambda \ne 0$ form a group of transformations. The map $\eta_\lambda : \Theta(N) \to \Theta(N)$ has inverse $\eta_{1/\lambda}$.

Also, from Theorem 1:

4.iii. For any $a_1, a_2 \in \Theta(N)$ the map $\lambda \to \mathrm{dist}(\eta_\lambda(a_1), \eta_\lambda(a_2))$ is a non-decreasing function of $|\lambda|$. In particular, for $\lambda \ge 1$ we have $\mathrm{dist}(\eta_\lambda(a_1), \eta_\lambda(a_2)) \ge \mathrm{dist}(a_1, a_2)$.

The following property is also clear since multiplication times λ is analytic in N,

and the exponential map is a diffeomorphism for Ω^ε.

4.iv. The map $(\lambda, a) \to \eta_\lambda(a)$ from $\mathbf{R} \times \Omega^\varepsilon$ to G^+ is smooth (in fact, analytic).

We get now to the main properties for these maps:

5. Proposition. *For $\lambda \geq 1$ and $a \in \Omega^\varepsilon$ the tangent map $T(\eta_\lambda)_a \to T(G^+)_{\eta_\lambda(a)}$ is a dilation for the Finsler metric, i.e.,*

$$\|T(\eta_\lambda)_a X\|_{\eta_\lambda(a)} \geq \|X\|_a$$

for any $X \in T(G^+)_a$.

To see this, let $\gamma(t)$ be a curve with $\gamma(0) = a$, $\dot\gamma(0) = X$. Then

$$\lim_{t\to 0^+} \frac{\operatorname{dist}(\gamma(t), a)}{t} = \|\dot\gamma(0)\|_a. \tag{5.i}$$

The general case follows from the case $a = 1$ by invariance under the group action. For $a = 1$ (5.i) can be proved using a $o(t)$ argument to the curve $\gamma(t) = e^{tX}$.

Using (5.i), the formula corresponding to (5.i) for $\eta_\lambda(a)$, and (4.iii) we get

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} \eta_\lambda(\gamma(t))\Big|_{t=0} \right\|_{\eta_\lambda(a)} \geq \|\dot\gamma(0)\|_a$$

for any $\lambda \geq 1$, and this proves Proposition 5.

6. Proposition. *For each $\lambda \neq 0$ and each $a \in \Omega^\varepsilon$, the tangent map $T(\eta_\lambda)_a : T(G^+)_a \to T(G^+)_{\eta_\lambda(a)}$ is a Banach space isomorphism (onto). In particular, $\Theta(N)$ is open in G^+ and $\Theta : N \to \Theta(N)$ is a diffeomorphism.*

Proof. Using the group action we may assume without loss of generality that $a = e^X$ with $X \in H^s$. Then $\eta_\lambda(a) = a^\lambda$. Since $\operatorname{dist}(a, B^+) < \varepsilon$ and the map η_λ is defined on the open set Ω^ε we can identify $T(\eta_\lambda)_a$ to an operator $\phi_\lambda : (A^s, \|\ \|_a) \to (A^s, \|\ \|_{a^\lambda})$.

The λ real for which ϕ_λ is an isomorphism (onto) form an open set that contains 1. Suppose that ϕ_λ is an isomorphism for each $1 \leq \lambda < m$. From Proposition 5 follows that $\|\phi_\lambda X\|_{a^\lambda} \geq \|X\|_a$ for all $X \in A^s$, or by definition of the Finsler metric,

$$\|a^{-\frac{\lambda}{2}} (\phi_\lambda X) a^{-\frac{\lambda}{2}}\| \geq \|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\|.$$

This implies that $\|\phi_\lambda^{-1} X\| \leq \|a\|\ \|a^{-\lambda}\|\ \|X\|$ for $1 \leq \lambda < m$. Hence ϕ_λ^{-1} is uniformly bounded (in the ordinary norm!) for $1 \leq \lambda < m$. Therefore $\phi_m = \lim_{\lambda\to m} \phi_\lambda$ is also an invertible operator because $\|\phi_m \phi_\lambda^{-1} - 1\| \leq \|\phi_m - \phi_\lambda\| \sup \|\phi_\lambda^{-1}\| \leq \varepsilon$ for λ close to m and then $\phi_m \phi_\lambda^{-1}$ is invertible, whence ϕ_m is also invertible. This shows that ϕ_λ is invertible for all $\lambda \geq 1$. The other cases of $\lambda \neq 0$ follow by routine arguments.

To complete the proof of Theorem 2 we need to show that $\Theta : N \to G^+$ is onto, and this in turn will follow from proving that $\Theta(N)$ is closed in G^+. Suppose then that $\{x_n\}$ is a Cauchy sequence in $\Theta(N)$. Using the fact that $\{x_n\}$ is bounded, choose $0 < \lambda < 1$ so small that $\operatorname{dist}(\eta_\lambda(x_n), B^+) < \delta$, where $W^\delta \subset \Omega^\varepsilon = \Theta(N^\varepsilon)$ according to Proposition 3.

Notice that by (4.iii)

$$\operatorname{dist}(\eta_\lambda(x_k), \eta_\lambda(x_m)) \geq \operatorname{dist}(x_k, x_m)$$

and therefore $\{\eta_\lambda(x_n)\}$ is Cauchy in G^+. But the tube $W^\delta = \{a \in G^+; \operatorname{dist}(a, B^+) \leq \delta\}$

is closed in G^+ and therefore $\lim_{n\to\infty} \eta_\lambda(x_n) = a_\infty$ is in Ω^ε, and a fortiori in $\Theta(N)$. Then $\lim x_n = \eta_{\lambda^{-1}}(a_\infty)$ is also in $\Theta(N)$, and $\Theta(N)$ is closed. The fact that Θ is a diffeomorphism is now an easy consequence of Proposition 6, and Theorem 2 is proved. □

7. Corollary. *The maps*

$$\Xi_1(b,c) = (bc)^+, \quad \Xi_2(b,c) = bcb$$

are analytic diffeomorphims from $B^+ \times C$ *onto* G^+.

Proof. In fact, each $\Theta(X) = \exp_b(X)$ is, by definition of the normal bundle, of the form $b^{\frac{1}{2}} c b^{\frac{1}{2}}$ for $c \in C$. Thus Theorem 2 applies to conclude that Ξ_2 is a diffeomorphism. The statement about Ξ_1 follows from the change of variable $a \to a^{-\frac{1}{2}}$. For details see (2.5) and (2.7) of [2]. □

We close this paper with the following theorem:

8. Theorem. *Denote by* U *the group of unitary elements of* A. *Then the map* $(b,c,u) \to bcu$ *is an analytic diffeomorphism of* $B^+ \times C \times U$ *onto the group* G *of all invertible elements of* A.

Proof. For $g \in G$, use the polar decomposition to write $g = g^+ v$ with $g^+ \in G^+$, $v \in U$ and Corollary 7 to write $g = (bc)^+ v$. For any $x \in G$, $v \in U$ we clearly have $(xv)^+ = x^+$ and therefore $g^+ = ((bc)^+ v)^+ = (bc)^+$ and then from (polar decomposition again) $(bc)^+ = (bc)w$ with $w \in U$ we get $g = bcu$ with $b \in B^+$, $c \in C$, and $u \in U$. Let now $bcu = b_1 c_1 u_1$. Then $(bc)^+ = (b_1 c_1)^+$ and Corollary 7 applies again to give $b = b_1$ and $c = c_1$. Uniqueness follows. □

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H. PORTA
Department of Mathematics
University of Illinois
Urbana
USA

L. RECHT
Departamento de Matemáticas
Universidad Simón Bolívar
Caracas
Venezuela

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