Emergent Gauge Symmetries and Quantum Operations

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- Notion of entanglement measure, the von Neumann entropy, for a general quantum system [1]

  Partial trace $\longleftrightarrow$ Restriction of a state to a subalgebra

- GNS representation provides an entropy, which in general is not unique [2]

  Purification $\longleftrightarrow$ Irreducible decomposition of GNS space

- Goal: Explain this ambiguity in terms of Tomita-Takesaki theory:
  - Emergent gauge symmetry
  - Quantum operations

[1] Balachandran, Govindarajan, de Queiroz, Reyes-Lega (2013)
[2] Balachandran, de Queiroz, Vaidya (2013)

Outline

Algebraic approach

- Observable algebra $\mathcal{A} \to C^*$-algebra.

We will consider only finite dimensional unital algebra.

- States $\omega : \mathcal{A} \to \mathbb{C}$ positive linear functional such that

$$\omega(\mathbb{1}_{\mathcal{A}}) = 1, \qquad \omega(a^*) = \overline{\omega(a)}, \qquad \forall a \in \mathcal{A}. \tag{1}$$

We will consider only faithful states, i.e $\omega(a^* a) = 0 \;\Rightarrow a = 0$.

GNS construction

$$(\mathcal{A}, \omega) \to (\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$$

- $\mathcal{A}$ has a vector space structure $a \to |a\rangle$.
- "Inner product" $a, b \in \mathcal{A}$

$$\omega(a^* b) := \langle a|b\rangle\,, \quad \omega(a^* a) = 0 \not\Rightarrow a = 0 \tag{2}$$

- Null space: Gelfand ideal

$$\mathcal{N}_\omega = \{a \in \mathcal{A} \mid \omega(a^* a) = 0\} \tag{3}$$

- GNS Hilbert space

$$\mathcal{H}_\omega = \overline{\mathcal{A}/\mathcal{N}_\omega}, \quad \omega(a^* b) = \langle [a]|[b]\rangle \tag{4}$$

ω faithful $\Rightarrow$ $\mathcal{N}_\omega$ trivial.

GNS construction

- Unique cyclic representation

$$\begin{aligned} \pi_\omega : \mathcal{A} &\to \mathcal{L}(\mathcal{H}_\omega) \\ a &\mapsto \pi_\omega(a) \quad : \quad \begin{aligned} \mathcal{H}_\omega &\to \mathcal{H}_\omega \\ \pi_\omega(a)\,|[b]\rangle &\mapsto |[ab]\rangle\,. \end{aligned} \end{aligned} \tag{5}$$

- Cyclic vector $|\Omega_\omega\rangle := |[\mathbb{1}_\mathcal{A}]\rangle$, i.e $\overline{\pi_\omega(\mathcal{A})\,|\Omega_\omega\rangle} = \mathcal{H}_\omega$ such that

$$\omega(a) = \langle \Omega_\omega | \pi_\omega(a) | \Omega_\omega \rangle\,, \qquad \forall a \in \mathcal{A}. \tag{6}$$

ω faithful $\Rightarrow$ π_ω faithful

ω faithful $\Rightarrow$ $|\Omega_\omega\rangle$ separating for $\pi_\omega(\mathcal{A})$

$$\pi_\omega(a)\,|\Omega_\omega\rangle = 0 \Rightarrow \pi_\omega(a) = 0, \quad a \in \mathcal{A}. \tag{7}$$

Finite dimensional case

Theorem (Takahashi 2003)

The structure theorem of finite dimensional C^*-algebras guarantees the existence of unique positive integers $n_1, \ldots, n_N$ such that

$$\mathcal{A} = \bigoplus_{r=1}^{N} \mathcal{A}_r, \qquad \mathcal{A}_r \cong M_{n_r}(\mathbb{C}), \quad 1 \leq r \leq N. \tag{8}$$

- $\mathbb{1}_{\mathcal{A}_r} \in \mathcal{A}$: orthogonal projection onto $\mathcal{A}_r$.
- The projector $P^r := \pi_\omega(\mathbb{1}_{\mathcal{A}_r})$ induces a (unique) decomposition of the GNS space into reducible subrepresentations

$$\mathcal{H}_\omega = \bigoplus_{r=1}^{N} \mathcal{H}^r, \qquad \mathcal{H}^r := P^r \mathcal{H}_\omega \tag{9}$$

Questions

- Each H^r can be further decomposed into irreducible subrepresentations with multiplicity n_r, but this is non-unique.

$$\mathcal{H}^r = \bigoplus_{k=1}^{n_r} \mathcal{H}^{(r,k)} \tag{10}$$

- Modular theory will characterize this decomposition $\rightarrow$ It will give rise to a non-abelian gauge symmetry.

- Emergent system **C**: $\mathcal{F} := \mathcal{L}(\mathcal{H}_\omega)$
- Subsystem **A**: $\mathcal{A} \simeq \pi_\omega(\mathcal{A}) \subseteq \mathcal{F}$.
- Purification **A** $\hookrightarrow$ **C**

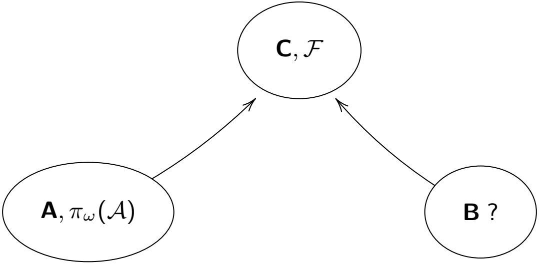
$$\langle \Omega_\omega | \pi_\omega(a) | \Omega_\omega \rangle = \langle \mathbb{1}_\mathcal{A} | a \rangle = \omega(a).$$

- Restriction of a state to a subalgebra is a generalization of partial trace [3].

[3] Balachandran, Govindarajan, de Queiroz, Reyes-Lega (2013)

Questions

System **B** $\hookrightarrow$ **C** complementary to **A**?



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Modular theory gives the answer.

Tomita-Takesaki Theory

- Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a C^*-algebra, $\mathcal{A}'$ its commutant

$$\mathcal{A}' = \{a \in \mathcal{L}(\mathcal{H}) \mid ab = ba, \ \forall b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}). \tag{12}$$

and $\Omega \in \mathcal{H}$ a cyclic and separating vector for $\mathcal{A}$ and $\mathcal{A}'$.

$\Omega \in \mathcal{H}$ cyclic for $\mathcal{A}$ $\leftrightarrow$ separating for $\mathcal{A}'$.

$\mathcal{A} = \mathcal{A}''$ von Neumann algebra.

Definition

The (antilinear) Tomita operator $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$\begin{aligned} S(a\Omega) &:= a^*\Omega, & a &\in \mathcal{A}, \\ S^*(a'\Omega) &:= (a')^*\Omega, & a' &\in \mathcal{A}'. \end{aligned} \tag{13}$$

Tomita operator and modular objects

- Polar decomposition of S

$$S = J\Delta^{1/2} = \Delta^{-1/2}J. \tag{14}$$

- Δ (unique) positive selfadjoint operator $\rightarrow$ the modular operator
- J (unique) antiunitary operator $\rightarrow$ the modular conjugation
- The vector Ω is invariant under

$$S\Omega = \Omega, \quad J\Omega = \Omega, \quad \Delta\Omega = \Omega. \tag{15}$$

- They satisfy

$$J = J^*, \quad J^2 = \mathbb{1}, \quad \Delta = S^* S. \tag{16}$$

Modular group

Definition

Let Δ be the modular operator, we construct a strongly continuous unitary group (via the functional calculus):

$$\Delta^{it} = \exp\big(it(\ln \Delta)\big), \quad t \in \mathbb{R}. \tag{17}$$

It is called the modular group and

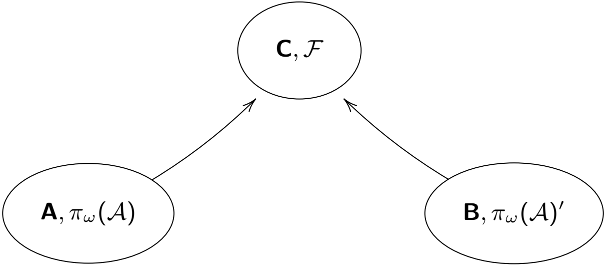
$$\sigma_t(a) := \Delta^{it} a \Delta^{-it}, \quad a \in \mathcal{A},\ t \in \mathbb{R}, \tag{18}$$

gives a one parameter automorphism group on $\mathcal{A}$, the so-called modular automorphism group.

Tomita-Takesaki Theorem

$$J\mathcal{A}J = \mathcal{A}' \qquad \text{and} \qquad \sigma_t(\mathcal{A}) = \mathcal{A}, \quad t \in \mathbb{R}. \tag{19}$$

Answer: Complementary system



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Answer: Irreducible decomposition → Gauge symmetry

- $e_{ij}^{(r)} \in \mathcal{A}_r$: matrix units

$$e_{ij}^{(r)} e_{k\ell}^{(s)} = \delta_{rs}\delta_{jk} e_{i\ell}^{(r)} \quad \left(e^{(r)}\right)_{ij}^{*} = e_{ji}^{(r)}, \quad \sum_{i=1}^{n_r} e_{ii}^{(r)} = \mathbb{1}_{\mathcal{A}_r}. \tag{21}$$

- $G \equiv U_{\mathcal{A}}$: group of unitary elements of $\mathcal{A}$.
- G acts on $\mathcal{H}_\omega$ via the representation

$$U(g) = J\pi_\omega(g)J \in \pi_\omega(\mathcal{A})', \qquad \forall g \in G. \tag{22}$$

- $U(G) = U_{\pi_\omega(\mathcal{A})'}$.

- $p_g^{(r,k)} := g\, e_{kk}^{(r)}\, g^*$: projectors

$$\mathcal{H}_g^{(r,k)} := P_g^{(r,k)} \mathcal{H}_\omega, \qquad P_g^{(r,k)} := J \pi_\omega(p_g^{(r,k)}) J \in \pi_\omega(\mathcal{A})'. \tag{23}$$

- $\sum_k P_g^{(r,k)} = \pi_\omega(\mathbb{1}_{\mathcal{A}}) = P_r$

$$\mathcal{H}^r = \bigoplus_{k=1}^{n_r} \mathcal{H}_g^{(r,k)}, \qquad H_g^{(r,k)} = \{U(g)|e_{ik}^{(r)}\rangle \mid i = 1, \ldots, n_r\}. \tag{24}$$

$$\mathcal{H}_\omega = \bigoplus_{r=1}^{N} \bigoplus_{k=1}^{n_r} \mathcal{H}_g^{(r,k)} \tag{25}$$

Gauge group

We regard G as a gauge group for $\mathcal{A}$

- $U(g) \in \mathbf{B} \Rightarrow$ its action will remain unnoticed, as far as system **A** is concerned.
- $\mathcal{F}$: algebra of fields [4]

$$\pi_\omega(\mathcal{A}) = \mathcal{F} \cap U(G)' = U(G)' \tag{26}$$

- Superselection sectors: G selects the observable algebra out of the field algebra.

[4] Doplicher, Haag, Roberts 1969

Sates and quantum operation

- Let $\mathcal{S}(\mathcal{F})$ be the set of states on $\mathcal{F}$, i.e

$$\begin{aligned}\mathcal{S}(\mathcal{F}) :=& \left\{ \varphi : \mathcal{F} \to \mathbb{C} \,\middle|\, \varphi \text{ is a state } \right\} \\ \equiv& \left\{ \rho_\varphi \in \mathcal{F} \,\middle|\, \varphi(f) = \mathrm{Tr}_{\mathcal{H}_\omega}(\rho_\varphi f),\ \forall f \in \mathcal{F} \right\}.\end{aligned} \tag{27}$$

  In particular $\rho \geq 0$ and $\mathrm{Tr}_{\mathcal{H}_\omega}(\rho) = 1$, i.e, is a density operator.
- For each φ, such ρ_φ is unique.
- $\mathbf{A} \hookrightarrow \mathbf{C}$

$$\mathcal{S}(\mathcal{F})\Big|_{\mathbf{A}} = \left\{ \rho \in \mathcal{S}(\mathcal{F}) \,\middle|\, \mathrm{Tr}_{\mathcal{H}_\omega}\left(\rho \pi_\omega(a)\right) = \omega(a),\ a \in \mathcal{A} \right\}$$

- In particular $|\Omega_\omega\rangle\langle\Omega_\omega| \in \mathcal{S}(\mathcal{F})\Big|_{\mathbf{A}}$.

Quantum operation

Proposition

Let $\rho \in \mathcal{S}(\mathcal{F})\Big|_{\mathbf{A}}$. *Then, the quantum operation*

$$\mathcal{E}_\Lambda(\rho) := \sum_k \Lambda_k\,\rho\,\Lambda_k^* \;\in\; \mathcal{S}(\mathcal{F})\Big|_{\mathbf{A}}, \qquad \sum_k \Lambda_k^*\Lambda_k = \mathbb{1}, \tag{28}$$

where the Kraus operators $\Lambda_k \in \pi_\omega(\mathcal{A})'$ *for all* k.

Proof.

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}_\omega}\left(\mathcal{E}_\Lambda(\rho)\pi_\omega(a)\right) &= \mathrm{Tr}_{\mathcal{H}_\omega}\left(\sum_k \Lambda_k\,\rho\,\Lambda_k^*\pi_\omega(a)\right) \\
&= \mathrm{Tr}_{\mathcal{H}_\omega}\left(\sum_k \Lambda_k^*\Lambda_k\,\rho\,\pi_\omega(a)\right) \\
&= \omega(a).
\end{aligned} \tag{29}$$

□

Particular case: Projective measurements

$$\Lambda_{r,k} \equiv P_g^{(r,k)}, \qquad P_g^{(r,k)}\mathcal{H}_\omega = \mathcal{H}_g^{(r,k)}. \tag{30}$$

- $\mathcal{E}_\Lambda \rightarrow \mathcal{E}_g$

$$\rho_g := \mathcal{E}_g\big(|\Omega_\omega\rangle\langle\Omega_\omega|\big). \tag{31}$$

- In particular, the operator $\rho_1 \equiv \rho_{\mathbb{1}_\mathcal{A}}$ for $N = 1$ was used in (Balachandran, et al 2013) in order to compute entanglement entropies arising from restrictions.

Entropy

$$\rho_g = \sum_{r=1}^{N} \sum_{k=1}^{n_r} P_g^{(r,k)} |\Omega_\omega\rangle \langle \Omega_\omega| P_g^{(r,k)} \tag{32}$$

- ρ_g have (non trivial) eigenvectors $P_g^{(r,k)}|\Omega\rangle$, with eigenvalues

$$\lambda_{r,k}(g) := \| P_g^{(r,k)} |\Omega\rangle \|^2. \tag{33}$$

- g-dependent von Neumann entropy

$$S(\rho_g) = -\sum_{r=1}^{N} \sum_{k=1}^{n_r} \lambda_{r,k}(g) \log \lambda_{r,k}(g). \tag{34}$$

The entropy ambiguity is parametrized by the group G.

Restrction to the complementary subsystem

- We have the following relation between the traces

$$\mathrm{tr}_{\mathcal{A}}(a) = \sum_{r=1}^{N} \frac{1}{n_r} \mathrm{Tr}_{\mathcal{H}^r}(\pi_\omega(a)) = \mathrm{tr}_{\pi_\omega(\mathcal{A})}(J\pi_\omega(a^*)J) = \mathrm{tr}_{\pi_\omega(\mathcal{A})}(\pi_\omega(a)), \quad (35)$$

- The faithful state $\omega : \mathcal{A} \to \mathbb{C}$ have associated a unique positive element $R \equiv R_\omega \in \mathcal{A}$ such that $\omega(a) = \mathrm{tr}_{\mathcal{A}}(Ra)$

Proposition

Let $\rho \in \mathcal{S}(\mathcal{F})\Big|_{\mathbf{A}}$ and $J\rho J = \rho$. Then, the unique density operator on $\pi_\omega(\mathcal{A})'$ implementing $\mathcal{E}_g(\rho)\Big|_{\mathbf{B}}$ is $\mathcal{E}_g(J\pi_\omega(R)J)$:

$$\mathrm{Tr}_{\mathcal{H}_\omega}(\mathcal{E}_g(\rho)b) = \mathrm{tr}_{\pi(\mathcal{A})'}\left(\mathcal{E}_g(J\pi_\omega(R)J)b\right), \qquad \forall b \in \mathbf{B}. \quad (36)$$

Relations between the entropies

Proposition

$$S(J\pi(R)J) = S(\rho_1) = S(R), \qquad S(\mathcal{E}_g(J\pi(R)J)) = S(\rho_g), \tag{37}$$

for all $g \in G \equiv U_{\mathcal{A}}$.

- Relative entropy

$$S(\mathcal{E}_g(J\pi_\omega(R)J)||J\pi_\omega(R)J) = S(\rho_g) - S(\rho_1) \equiv \Delta S \geq 0. \tag{38}$$

- Then

$$S(\rho_g) \geq S(\rho_1) \tag{39}$$

- In particular

$$S(\rho_g|_{\mathbf{A}}) = S(\rho_1), \qquad S(\rho_g|_{\mathbf{B}}) = S(\rho_g). \tag{40}$$

- G:gauge group for system **A**.
- Ambiguity in the entropy is carried by system **B**.

Example: Bipartite system

- $N = 1$, then $\mathcal{A} = M_n(\mathbb{C})$, with matrix units e_{ij}.
- $\omega(a) = \mathrm{tr}_{\mathcal{A}}(Ra)$,

$$R = \sum_i \lambda_i e_{ii}, \qquad \lambda_i > 0. \tag{41}$$

- The vectors

$$|\hat{e}_{ij}\rangle := (\lambda_j)^{-1/2} |e_{ij}\rangle \tag{42}$$

provides an orthonormal basis for $\mathcal{H}_\omega$.

- Hilbert space isomorphism

$$\begin{aligned} \Phi : \mathcal{H}_\omega &\rightarrow \mathbb{C}^n \otimes \mathbb{C}^n \\ |\hat{e}_{ij}\rangle &\mapsto \Phi(|\hat{e}_{ij}\rangle) := |i\rangle \otimes |j\rangle. \end{aligned} \tag{43}$$

- $T \in \mathcal{L}(\mathcal{H}_\omega) \ \rightarrow \ \widetilde{T} := \Phi T \Phi^{-1} \in \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^n)$.

- $\widetilde{J}(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle$.
- $\widetilde{\pi}_\omega(a) = a \otimes \mathbb{1}_n, \quad a \in \mathcal{A}$.
- $\widetilde{J}\widetilde{\pi}_\omega(a)\widetilde{J} = \mathbb{1}_n \otimes \bar{a}, \quad a \in \mathcal{A}$.

$$\begin{aligned} \mathbf{A} &\rightarrow \mathcal{A} \otimes \mathbb{1}_n \\ \mathbf{B} &\rightarrow \mathbb{1}_n \otimes \mathcal{A} \\ \mathbf{C} &\rightarrow \mathcal{A} \otimes \mathcal{A} \end{aligned} \tag{44}$$

- For $g \in G \equiv U(n)$,

$$\widetilde{\rho}_g|_{\mathbf{B}} = \sum_k \lambda_k(g)\, \bar{g}|k\rangle\langle k|\bar{g}^*, \qquad \lambda_k(g) := \sum_i \lambda_i |g_{ik}|^2. \tag{45}$$

- Its restriction to the system **B** is given by

$$\widetilde{\mathcal{E}}_g(\widetilde{J}\widetilde{\pi}_\omega(R)\widetilde{J}) = \mathbb{1}_n \otimes (\widetilde{\rho}_g|_{\mathbf{B}}). \tag{46}$$

Work in progress: Ethylene molecule $C_2 H4$

- Homogeneous spaces

$$\mathcal{Q} = G/H \tag{47}$$

G: compact Lie group, H: non- abelian finite subgroup

- The configuration space [5]

$$\mathcal{Q} = SO(3)/H \sim SU(2)/H^* \tag{48}$$

H dihedral group: gauge group

- Quantization on $T^*\mathcal{Q} \rightarrow C^*(G \times Q)$: covariance algebra.
- Evolution: Casimir $\leftrightarrow$ Modular operator.
- Hamiltonian formalism $\rightarrow$ anomalies.[6]

[5] Balachandran, de Queiroz, Vaidya (2013)
[6] Esteve, 1986

Conclusions

- We construct a canonical embedding and purification of a quantum system by means of the GNS construction and we identify a subsystem decomposition using modular theory.
- We identify a gauge symmetry in the sense of DHR.
- We define a family of entropy-increasing quantum operations induced by gauge transformations, which leave invariant the original system.

Gauge symmetries $\leftrightarrow$ quantum operations

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