Majorana Fermions and Orthogonal Complex Structures

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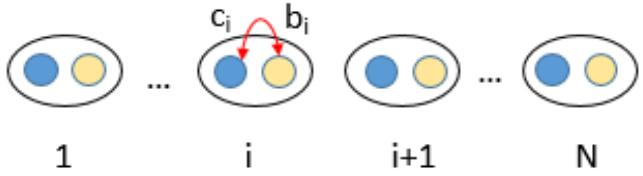
Ground states of quadratic Hamiltonians for fermionic systems can be characterized in terms of orthogonal complex structures. The standard way in which such Hamiltonians are diagonalized makes use of a certain "doubling" of the Hilbert space. In this work we show that this redundancy in the Hilbert space can be completely lifted if the relevant orthogonal structure is taken into account. Such an approach allows for a treatment of Majorana fermions which is both physically and mathematically transparent. Furthermore, an explicit connection between orthogonal complex structures and the topological $\mathbb{Z}_2$-invariant is given.



- Pauli exclusion
- Inequivalent representations in the quantum limit.

Given a 1-particle Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, the **CAR algebra** $\mathcal{A}_{\mathrm{CAR}}(\mathcal{H}, \langle \cdot, \cdot \rangle)$, is generated by $a(u)$ and $a^\dagger(u)$ (for $u \in \mathcal{H}$), satisfying:

$$\{a(u), a^\dagger(v)\} = \langle u, v \rangle$$
$$\{a(u), a(v)\} = \{a^\dagger(u), a^\dagger(v)\} = 0.$$

Consider a fermionic Hamiltonian of the form

$$H = \sum_{i,j=1}^{N} \left[a_i^\dagger A_{ij} a_j + \frac{1}{2} \left(a_i^\dagger B_{ij} a_j^\dagger + a_i \overline{B}_{ij} a_j \right) \right]$$

It can be written in the form

$$H = \frac{1}{2} (a^\dagger, a) \begin{pmatrix} A & B \\ -\overline{B} & -\overline{A} \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \tag{1}$$

- Bogoliubov and Araki for diagonalization.

Set $(V, g(\cdot,\cdot))$ a real vector space with $\dim_{\mathbb{R}}(V) = 2n$ with a positive, symmetric bilinear form.
An **Orthogonal Complex Structure (OCS)** is a real linear operator $J : V \to V$ such that $J^2 = -1$ and $g(Ju, Jv) = g(u, v)$

- We want a complexification of V different from the usual one $V^{\mathbb{C}}$
- $\mathbb{C}l(V)$ acts on the Fock space but its representation is not irreducible.

The **complex vector space** V_J is V with complex multiplication given by $iv := Jv$.
Define an inner product in V_J by

$$\langle u, v \rangle_J := g(u, v) + ig(Ju, v), \tag{2}$$

and we obtain a complex Hilbert space $(V_J, \langle \cdot, \cdot \rangle_J)$ with $\dim_{\mathbb{C}}(V_J) = n$.

Given J an OCS, for $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ define the **Polarization**:

$$P_{\pm J} := \frac{1}{2}(1 \mp iJ),$$

and $W_{\pm J} := P_{\pm J}(V^{\mathbb{C}})$. Then we have

$$\boxed{V^{\mathbb{C}} = W_J \oplus W_{-J} = W_J \oplus W_J^{\perp}} \tag{3}$$

We obtain the following unitary isomorphism:

$$\boxed{(V_J, \langle \cdot, \cdot \rangle_J) \cong (W_J, \langle\langle \cdot, \cdot \rangle\rangle)} \tag{4}$$

One-to-one correspondence between polarizations and OCS.
This is for every W there exists J_W such that:
$W = \{u - iJ_W u : u \in \mathcal{H}\}$

- There is a relation between the action $W \mapsto hW$ and its dual action $J_W \mapsto hJ_W h^{-1}$ with $h \in O(\mathcal{H})$
- $\mathcal{J}$, the space of OCS is characterized by:

$$\boxed{\mathcal{J} = O(2n)/U(n)} \tag{5}$$

- Choose the correct J for diagonalization.

$(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ a complex Hilbert space together with a **conjugation** Γ, that is, an anti-unitary operator in $\mathcal{K}$ such that $\Gamma^2 = 1$. The *self-dual* CAR algebra $\mathcal{A}^{\mathrm{sd}}_{\mathrm{CAR}}(\mathcal{K}, \Gamma)$ is the $*$-algebra generated by the identity and operators $B^\dagger(u)$, $B(v)$:

$$\{B(u), B^\dagger(v)\} = \langle u, v \rangle_{\mathcal{K}}\,, \tag{6}$$

$$B^\dagger(\lambda u + \mu v) = \lambda B^\dagger(u) + \mu B^\dagger(v), \tag{7}$$

$$B^\dagger(u) = B(\Gamma u). \tag{8}$$

Theorem: If $\mathcal{K}$ is either even or infinite dimensional there exists a projection operator E such that $\Gamma\, E\, \Gamma = 1 - E$ and there is an algebra isomorphism

$$\boxed{\mathcal{A}^{\mathrm{sd}}_{\mathrm{CAR}}(\mathcal{K}, \Gamma) \cong \mathcal{A}_{\mathrm{CAR}}(E\mathcal{K})} \tag{9}$$

Isomorphims

Set $\mathcal{K} := \mathcal{H} \oplus \overline{\mathcal{H}}$ with $x \in \mathcal{K}$ written as $x = (x_1, x_2)$. Consider the projection $Px := (x_1, 0)$ and the complex conjugation $\Gamma(x) := (x_2, x_1)$. Define the *real subspace*

$$\mathrm{Re}_\Gamma \mathcal{K} := \{x \in \mathcal{K} : \Gamma(x) = x\}$$

Let $\mathcal{H} = V_J$. Endow $\mathcal{H} \oplus \overline{\mathcal{H}}$ with the complex conjugation defined above. Then we have the following equivalences:

$$\boxed{\mathbb{C}\ell(V) \cong \mathcal{A}_{\mathrm{CAR}}(\mathcal{H}) \cong \mathcal{A}^{\mathrm{sd}}_{\mathrm{CAR}}(\mathcal{H} \oplus \overline{\mathcal{H}}, \Gamma)} \tag{10}$$

- No redundancy. A direct link between (i) the Bogoliubov transformation that diagonalizes H (ii) the corresponding orthogonal complex structure and (iii) the topological $\mathbb{Z}_2$ index.

$\mathbb{Z}_2$ index

Let $h \in O(V, g)$ be such that the Hamiltonian becomes diagonal in the fermionic operators

$$c(v) = a(p_h) + a^\dagger(q_h v), \tag{11}$$

where $p_h = (h - JhJ)/2$ and $q_h = (h + JhJ)/2$ (Bogoliubov transformation). The ground state of H is characterized by the OCS $J_h = hJh^{-1}$. The map

$$\begin{aligned} \text{index} : \mathcal{J} &\longrightarrow \mathbb{Z}_2 \\ J_h &\longmapsto (-1)^{\frac{1}{2}\dim\ker(J+J_h)} \end{aligned} \tag{12}$$

gives exactly the topological $\mathbb{Z}_2$ index (Pfaffian invariant).

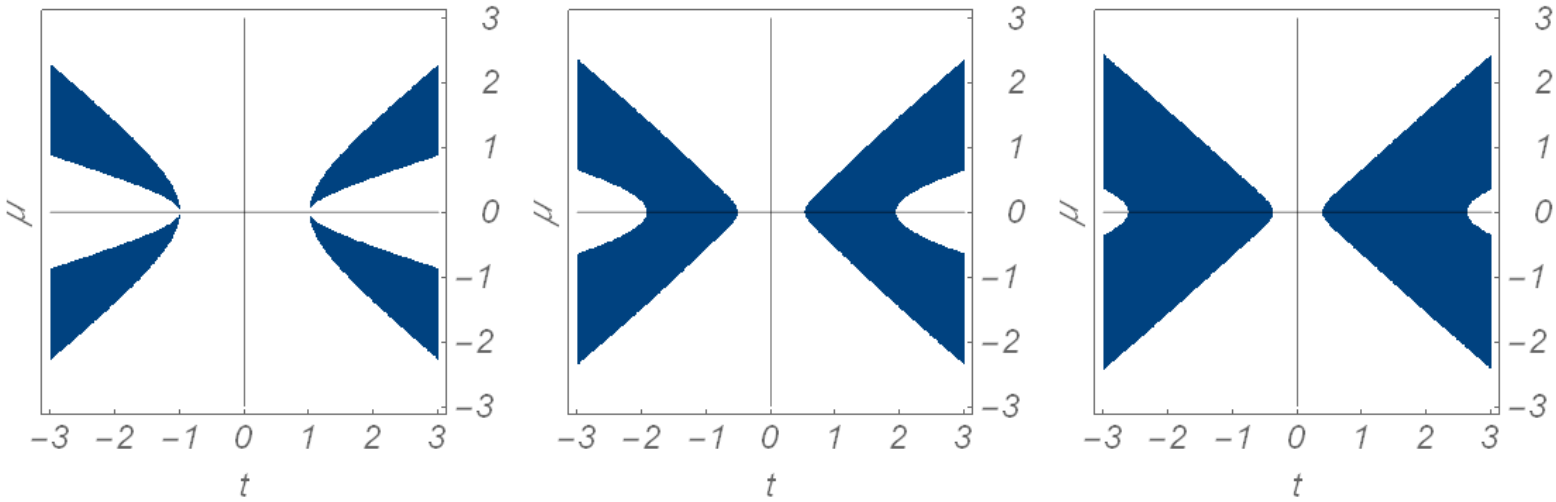
$$\begin{aligned} H =& t(a_1^\dagger a_2 + a_2^\dagger a_1) + \Delta(a_1^\dagger a_2^\dagger - a_1 a_2) \\ & - 2\mu(a_1^\dagger a_1 + a_2^\dagger a_2). \end{aligned}$$

$\alpha = \sqrt{\Delta^2 + 4\mu^2}$, $\beta_\pm = \sqrt{(\alpha \pm \Delta)/(2\alpha)}$ and $\sigma = \mathrm{sgn}(\alpha - t)$. For the orthogonal complex structure we obtain

$$J_h = \frac{1}{\sqrt{\Delta^2 + 4\mu^2}} \begin{pmatrix} 0 & 0 & -2\sigma\mu & \Delta \\ 0 & 0 & -\sigma\Delta & -2\mu \\ 2\sigma\mu & \sigma\Delta & 0 & 0 \\ -\Delta & 2\mu & 0 & 0 \end{pmatrix}.$$

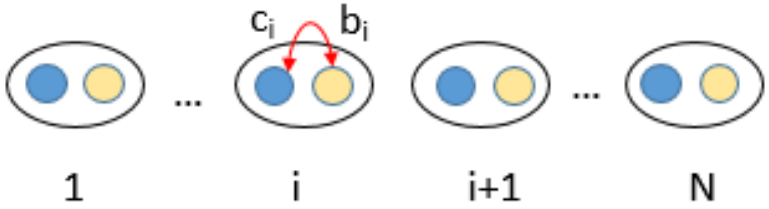
Finally, the $\mathbb{Z}_2$-index is given by

$$\text{index}(h) := (-1)^{\frac{1}{2}\dim\ker(J+J_h)} = \det h = \sigma. \tag{13}$$



(a) $N = 4, r = 0$ (b) $N = 4, r = 0.1$ (c) $N = 4, r = 0.2$

Figure: The shaded regions correspond to nontrivial values of the $\mathbb{Z}_2$-index associated to the Kitaev chain Hamiltonian. N denotes the number of sites and r is a parameter that allows us to interpolate between open boundary conditions ($r = 0$) and periodic ones ($r = 1$).



A fermionic operator f can be regarded as a superposition of two Majorana fermions: $f = \gamma^{(A)} + i\gamma^{(B)}$. We obtain the following characterization of the Majorana operators, in terms of the Clifford algebra generators:

$$\gamma^{(A)}(v) = \frac{1}{2}\pi_J(v), \quad \gamma^{(B)}(v) = \frac{1}{2}\pi_J(Jv). \tag{14}$$

Conclusions

1. OCS produces a reduction in the dimension of the Hilbert space, accounting for the "redundancy" of Hamiltonians of the BdG form.
2. The $\mathbb{Z}_2$-invariant is related to the OCS. Our formalism allows for a direct computation of the $\mathbb{Z}_2$-invariant independently of the boundary conditions chosen.